Chapter 9

Electromagnetic Waves

Problem 9.1

$$\begin{aligned}
\frac{\partial f_1}{\partial z} &= -2Ab(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right];\\
\frac{\partial f_1}{\partial t} &= 2Abv(z-vt)e^{-b(z-vt)^2};\ \frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2 e^{-b(z-vt)^2}\right] = v^2\frac{\partial^2 f_1}{\partial z^2}.\ \checkmark\\
\frac{\partial f_2}{\partial z} &= Ab\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)];\\
\frac{\partial f_2}{\partial t} &= -Abv\cos[b(z-vt)];\ \frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin[b(z-vt)] = v^2\frac{\partial^2 f_2}{\partial z^2}.\ \checkmark\\
\frac{\partial f_3}{\partial z} &= \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3};\\
\frac{\partial f_3}{\partial t} &= \frac{2Abv(z-vt)}{[b(z-vt)^2+1]^2};\ \frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3} = v^2\frac{\partial^2 f_3}{\partial z^2}.\ \checkmark\\
\frac{\partial f_4}{\partial z} &= -2Ab^2ze^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2e^{-b(bz^2+vt)}\right];\\
\frac{\partial f_4}{\partial t} &= -Abve^{-b(bz^2+vt)};\ \frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)} \neq v^2\frac{\partial^2 f_4}{\partial z^2}.\\
\frac{\partial f_5}{\partial z} &= Ab\cos(bz)\cos(bvt)^3;\ \frac{\partial^2 f_5}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3;\ \frac{\partial f_5}{\partial t} = -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3;\\
\frac{\partial^2 f_5}{\partial t^2} &= -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9Ab^6v^6t^4\sin(bz)\cos(bvt)^3 \neq v^2\frac{\partial^2 f_5}{\partial z^2}.
\end{aligned}$$

Problem 9.2

$$\begin{aligned}
\frac{\partial f}{\partial z} &= Ak\cos(kz)\cos(kvt);\ \frac{\partial^2 f}{\partial z^2} = -Ak^2\sin(kz)\cos(kvt);\\
\frac{\partial f}{\partial t} &= -Akv\sin(kz)\sin(kvt);\ \frac{\partial^2 f}{\partial t^2} = -Ak^2v^2\sin(kz)\cos(kvt) = v^2\frac{\partial^2 f}{\partial z^2}.\ \checkmark
\end{aligned}$$

Use the trig identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ to write

$$f = \boxed{\frac{A}{2}\left\{\sin[k(z+vt)] + \sin[k(z-vt)]\right\}},$$

which is of the form 9.6, with $g = (A/2)\sin[k(z - vt)]$ and $h = (A/2)\sin[k(z + vt)]$.

Problem 9.3

$$\begin{aligned}
(A_3)^2 &= \left(A_3 e^{i\delta_3}\right)\left(A_3 e^{-i\delta_3}\right) = \left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right)\left(A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2}\right) \\
&= (A_1)^2 + (A_2)^2 + A_1A_2\left(e^{i\delta_1}e^{-i\delta_2} + e^{-i\delta_1}e^{i\delta_2}\right) = (A_1)^2 + (A_2)^2 + A_1A_2 2\cos(\delta_1 - \delta_2); \\
A_3 &= \boxed{\sqrt{(A_1)^2 + (A_2)^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}.} \\
A_3 e^{i\delta_3} &= A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2) \\
&= (A_1\cos\delta_1 + A_2\cos\delta_2) + i(A_1\sin\delta_1 + A_2\sin\delta_2). \quad \tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}; \\
\delta_3 &= \boxed{\tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right).}
\end{aligned}$$

Problem 9.4

The wave equation (Eq. 9.2) says $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$. Look for solutions of the form $f(z,t) = Z(z)T(t)$. Plug this in: $T\dfrac{d^2Z}{dz^2} = \dfrac{1}{v^2}Z\dfrac{d^2T}{dt^2}$. Divide by ZT: $\dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = \dfrac{1}{v^2T}\dfrac{d^2T}{dt^2}$. The left side depends only on z, and the right side only on t, so both must be constant. Call the constant $-k^2$.

$$\left\{\begin{aligned}
\frac{d^2Z}{dz^2} &= -k^2 Z \quad \Rightarrow \quad Z(z) = Ae^{ikz} + Be^{-ikz}, \\
\frac{d^2T}{dt^2} &= -(kv)^2 T \quad \Rightarrow \quad T(t) = Ce^{ikvt} + De^{-ikvt}.
\end{aligned}\right\}$$

(Note that k must be *real*, else Z and T blow up; with no loss of generality we can assume k is *positive.*)
$f(z,t) = \left(Ae^{ikz} + Be^{-ikz}\right)\left(Ce^{ikvt} + De^{-ikvt}\right) = A_1e^{i(kz+kvt)} + A_2e^{i(kz-kvt)} + A_3e^{i(-kz+kvt)} + A_4e^{i(-kz-kvt)}$.
The general linear combination of separable solutions is therefore

$$f(z,t) = \int_0^\infty \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)} + A_3(k)e^{i(-kz+\omega t)} + A_4(k)e^{i(-kz-\omega t)}\right] dk,$$

where $\omega \equiv kv$. But we can combine the third term with the first, by allowing k to run *negative* ($\omega = |k|v$ remains positive); likewise the second and the fourth:

$$f(z,t) = \int_{-\infty}^\infty \left[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)}\right] dk.$$

Because (in the end) we shall only want the the *real part* of f, it suffices to keep only *one* of these terms (since k goes negative, both terms include waves traveling in both directions); the second is traditional (though either would do). Specifically,

$$\mathrm{Re}(f) = \int_{-\infty}^\infty [\mathrm{Re}(A_1)\cos(kz+\omega t) - \mathrm{Im}(A_1)\sin(kz+\omega t) + \mathrm{Re}(A_2)\cos(kz-\omega t) - \mathrm{Im}(A_2)\sin(kz-\omega t)]\, dk.$$

The first term, $\cos(kz+\omega t) = \cos(-kz-\omega t)$, combines with the third, $\cos(kz-\omega t)$, since the negative k is picked up in the other half of the range of integration, and the second, $\sin(kz+\omega t) = -\sin(-kz-\omega t)$, combines with the fourth for the same reason. So the general solution, for our purposes, can be written in the form

$$\tilde{f}(z,t) = \int_{-\infty}^\infty \tilde{A}(k)e^{i(kz-\omega t)}\, dk \quad \text{qed (the tildes remind us that we want the real part).}$$

Problem 9.5

Equation 9.26 $\Rightarrow g_I(-v_1t) + h_R(v_1t) = g_T(-v_2t)$. Now $\dfrac{\partial g_I}{\partial z} = -\dfrac{1}{v_1}\dfrac{\partial g_I}{\partial t}$; $\dfrac{\partial h_R}{\partial z} = \dfrac{1}{v_1}\dfrac{\partial h_R}{\partial t}$; $\dfrac{\partial g_T}{\partial z} = -\dfrac{1}{v_2}\dfrac{\partial g_T}{\partial t}$. Equation 9.27 $\Rightarrow -\dfrac{1}{v_1}\dfrac{\partial g_I(-v_1t)}{\partial t} + \dfrac{1}{v_1}\dfrac{\partial h_R(v_1t)}{\partial t} = -\dfrac{1}{v_2}\dfrac{\partial g_T(-v_2t)}{\partial t} \Rightarrow g_I(-v_1t) - h_R(v_1t) = \dfrac{v_1}{v_2}g_T(-v_2t) + \kappa$ (where κ is a constant).

Adding these equations, we get $2g_I(-v_1t) = \left(1 + \dfrac{v_1}{v_2}\right) g_T(-v_2t) + \kappa$, or $g_T(-v_2t) = \left(\dfrac{2v_2}{v_1 + v_2}\right) g_I(-v_1t) + \kappa'$ (where $\kappa' \equiv -\kappa\dfrac{v_2}{v_1 + v_2}$). Now $g_I(z,t)$, $g_T(z,t)$, and $h_R(z,t)$ are each functions of a single variable u (in the first case $u = z - v_1t$, in the second $u = z - v_2t$, and in the third $u = z + v_1t$). Thus

$$\boxed{g_T(u) = \left(\frac{2v_2}{v_1 + v_2}\right) g_I(v_1u/v_2) + \kappa'.}$$

Multiplying the first equation by v_1/v_2 and subtracting, $\left(1 - \dfrac{v_1}{v_2}\right) g_I(-v_1t) - \left(1 + \dfrac{v_1}{v_2}\right) h_R(v_1t) = \kappa \Rightarrow$ $h_R(v_1t) = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right) g_I(-v_1t) - \kappa\left(\dfrac{v_2}{v_1 + v_2}\right)$, or $\boxed{h_R(u) = \left(\dfrac{v_2 - v_1}{v_1 + v_2}\right) g_I(-u) + \kappa'.}$

[The notation is tricky, so here's an example: for a sinusoidal wave,

$$\begin{cases} g_I &= A_I\cos(k_1z - \omega t) &= A_I\cos[k_1(z - v_1t)] &\Rightarrow g_I(u) = A_I\cos(k_1u). \\ g_T &= A_T\cos(k_2z - \omega t) &= A_T\cos[k_2(z - v_2t)] &\Rightarrow g_T(u) = A_T\cos(k_2u). \\ h_R &= A_R\cos(-k_1z - \omega t) &= A_R\cos[-k_1(z + v_1t)] &\Rightarrow h_R(u) = A_R\cos(-k_1u). \end{cases}$$

Here $\kappa' = 0$, and the boundary conditions say $\dfrac{A_T}{A_I} = \dfrac{2v_2}{v_1 + v_2}$, $\dfrac{A_R}{A_I} = \dfrac{v_2 - v_1}{v_1 + v_2}$ (same as Eq. 9.32), and $\dfrac{v_1}{v^2}k_1 = k_2$ (consistent with Eq. 9.24).]

Problem 9.6

(a) $T\sin\theta_+ - T\sin\theta_- = ma \Rightarrow \boxed{T\left(\left.\dfrac{\partial f}{\partial z}\right|_{0+} - \left.\dfrac{\partial f}{\partial z}\right|_{0-}\right) = m\left.\dfrac{\partial^2 f}{\partial t^2}\right|_0.}$

(b) $\tilde{A}_I + \tilde{A}_R = \tilde{A}_T$; $T[ik_2\tilde{A}_T - ik_1(\tilde{A}_I - \tilde{A}_R)] = m(-\omega^2\tilde{A}_T)$, or $k_1(\tilde{A}_I - \tilde{A}_R) = \left(k_2 - \dfrac{im\omega^2}{T}\right)\tilde{A}_T$.

Multiply first equation by k_1 and add: $2k_1\tilde{A}_I = \left(k_1 + k_2 - i\dfrac{m\omega^2}{T}\right)\tilde{A}_T$, or $\tilde{A}_T = \left(\dfrac{2k_1}{k_1 + k_2 - im\omega^2/T}\right)\tilde{A}_I$.

$$\tilde{A}_R = \tilde{A}_T - \tilde{A}_I = \frac{2k_1 - (k_1 + k_2 - im\omega^2/T)}{k_1 + k_2 - im\omega^2/T}\tilde{A}_I = \left(\frac{k_1 - k_2 + im\omega^2/T}{k_1 + k_2 - im\omega^2/T}\right)\tilde{A}_I.$$

If the second string is massless, so $v_2 = \sqrt{T/\mu_2} = \infty$, then $k_2/k_1 = 0$, and we have $\tilde{A}_T = \left(\dfrac{2}{1 - i\beta}\right)\tilde{A}_I$, $\tilde{A}_R = \left(\dfrac{1 + i\beta}{1 - i\beta}\right)\tilde{A}_I$, where $\beta \equiv \dfrac{m\omega^2}{k_1T} = \dfrac{m(k_1v_1)^2}{k_1T} = \dfrac{mk_1}{T}\dfrac{T}{\mu_1}$, or $\boxed{\beta = m\dfrac{k_1}{\mu_1}.}$ Now $\left(\dfrac{1 + i\beta}{1 - i\beta}\right) = Ae^{i\phi}$, with $A^2 = \left(\dfrac{1 + i\beta}{1 - i\beta}\right)\left(\dfrac{1 - i\beta}{1 + i\beta}\right) = 1 \Rightarrow A = 1$, and $e^{i\phi} = \dfrac{(1 + i\beta)^2}{(1 - i\beta)(1 + i\beta)} = \dfrac{1 + 2i\beta - \beta^2}{1 + \beta^2} \Rightarrow$ $\tan\phi = \dfrac{2\beta}{1 - \beta^2}$. Thus $A_Re^{i\delta_R} = e^{i\phi}A_Ie^{i\delta_I} \Rightarrow \boxed{A_R = A_I,}$ $\boxed{\delta_R = \delta_I + \tan^{-1}\left(\dfrac{2\beta}{1 - \beta^2}\right).}$

Similarly, $\left(\dfrac{2}{1 - i\beta}\right) = Ae^{i\phi} \Rightarrow A^2 = \left(\dfrac{2}{1 - i\beta}\right)\left(\dfrac{2}{1 + i\beta}\right) = \dfrac{4}{1 + \beta^2} \Rightarrow A = \dfrac{2}{\sqrt{1 + \beta^2}}$.

$Ae^{i\phi} = \dfrac{2(1+i\beta)}{(1-i\beta)(1+i\beta)} = \dfrac{2(1+i\beta)}{(1+\beta^2)} \Rightarrow \tan\phi = \beta$. So $A_T e^{i\delta_T} = \dfrac{2}{\sqrt{1+\beta^2}} e^{i\phi} A_I e^{i\delta_I}$;

$\boxed{A_T = \dfrac{2}{\sqrt{1+\beta^2}} A_I;}$ $\boxed{\delta_T = \delta_I + \tan^{-1}\beta.}$

Problem 9.7

(a) $F = T\dfrac{\partial^2 f}{\partial z^2}\Delta z - \gamma\dfrac{\partial f}{\partial t}\Delta z = \mu\Delta z\dfrac{\partial^2 f}{\partial t^2}$, or $\boxed{T\dfrac{\partial^2 f}{\partial z^2} = \mu\dfrac{\partial^2 f}{\partial t^2} + \gamma\dfrac{\partial f}{\partial t}.}$

(b) Let $\tilde{f}(z,t) = \tilde{F}(z)e^{-i\omega t}$; then $Te^{-i\omega t}\dfrac{d^2\tilde{F}}{dz^2} = \mu(-\omega^2)\tilde{F}e^{-i\omega t} + \gamma(-i\omega)\tilde{F}e^{-i\omega t} \Rightarrow$

$T\dfrac{d^2\tilde{F}}{dz^2} = -\omega(\mu\omega + i\gamma)\tilde{F}$, $\dfrac{d^2\tilde{F}}{dz^2} = -\tilde{k}^2\tilde{F}$, where $\tilde{k}^2 \equiv \dfrac{\omega}{T}(\mu\omega + i\gamma)$. Solution : $\tilde{F}(z) = \tilde{A}e^{i\tilde{k}z} + \tilde{B}e^{-i\tilde{k}z}$.

Resolve $\tilde{k}$ into its real and imaginary parts: $\tilde{k} = k + i\kappa \Rightarrow \tilde{k}^2 = k^2 - \kappa^2 + 2ik\kappa = \dfrac{\omega}{T}(\mu\omega + i\gamma)$.

$2k\kappa = \dfrac{\omega\gamma}{T} \Rightarrow \kappa = \dfrac{\omega\gamma}{2kT}$; $k^2 - \kappa^2 = k^2 - \left(\dfrac{\omega\gamma}{2T}\right)^2\dfrac{1}{k^2} = \dfrac{\mu\omega^2}{T}$; or $k^4 - k^2(\mu\omega^2/T) - (\omega\gamma/2T)^2 = 0 \Rightarrow$

$k^2 = \dfrac{1}{2}\left[(\mu\omega^2/T) \pm \sqrt{(\mu\omega^2/T)^2 + 4(\omega\gamma/2T)^2}\right] = \dfrac{\mu\omega^2}{2T}\left[1 \pm \sqrt{1+(\gamma/\mu\omega)^2}\right]$. But k is real, so k^2 is positive, so we need the plus sign: $k = \omega\sqrt{\dfrac{\mu}{2T}}\sqrt{1+\sqrt{1+(\gamma/\mu\omega)^2}}$. $\kappa = \dfrac{\omega\gamma}{2kT} = \dfrac{\gamma}{\sqrt{2T\mu}}\left[1+\sqrt{1+(\gamma/\mu\omega)^2}\right]^{-1/2}$.

Plugging this in, $\tilde{F} = Ae^{i(k+i\kappa)z} + Be^{-i(k+i\kappa)z} = Ae^{-\kappa z}e^{ikz} + Be^{\kappa z}e^{-ikz}$. But the B term gives an exponentially *increasing* function, which we don't want (I assume the waves are propagating in the $+z$ direction), so $B = 0$, and the solution is $\boxed{\tilde{f}(z,t) = \tilde{A}e^{-\kappa z}e^{i(kz-\omega t)}.}$ (The actual displacement of the string is the real part of this, of course.)

(c) The wave is attenuated by the factor $e^{-\kappa z}$, which becomes $1/e$ when

$z = \dfrac{1}{\kappa} = \boxed{\dfrac{\sqrt{2T\mu}}{\gamma}\sqrt{1+\sqrt{1+(\gamma/\mu\omega)^2}};}$ this is the characteristic penetration depth.

(d) This is the same as before, except that $k_2 \to k + i\kappa$. From Eq. 9.29, $\tilde{A}_R = \left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\tilde{A}_I$;

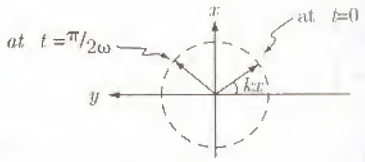
$\left(\dfrac{A_R}{A_I}\right)^2 = \left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right)\left(\dfrac{k_1 - k + i\kappa}{k_1 + k - i\kappa}\right) = \dfrac{(k_1-k)^2 + \kappa^2}{(k_1+k)^2 + \kappa^2}$. $\boxed{A_R = \sqrt{\dfrac{(k_1-k)^2+\kappa^2}{(k_1+k)^2+\kappa^2}}A_I}$

(where $k_1 = \omega/v_1 = \omega\sqrt{\mu_1/T}$, while k and κ are defined in part b). Meanwhile

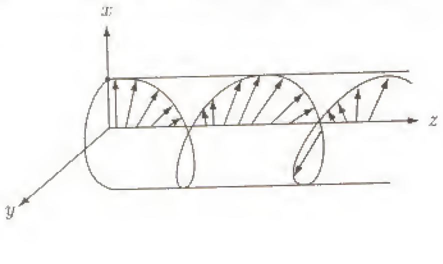
$\left(\dfrac{k_1 - k - i\kappa}{k_1 + k + i\kappa}\right) = \dfrac{(k_1 - k - i\kappa)(k_1 + k + i\kappa)}{(k_1+k)^2 + \kappa^2} = \dfrac{(k_1)^2 - k^2 - \kappa^2 - 2i\kappa k_1}{(k_1+k)^2 + \kappa^2} \Rightarrow \boxed{\delta_R = \tan^{-1}\left(\dfrac{-2k_1\kappa}{(k_1)^2 - k^2 - \kappa^2}\right).}$

Problem 9.8

(a) $\mathbf{f}_v(z,t) = A\cos(kz - \omega t)\,\hat{\mathbf{x}}$; $\mathbf{f}_h(z,t) = A\cos(kz - \omega t + 90°)\,\hat{\mathbf{y}} = -A\sin(kz - \omega t)\,\hat{\mathbf{y}}$. Since $f_v^2 + f_h^2 = A^2$, the vector sum $\mathbf{f} = \mathbf{f}_v + \mathbf{f}_h$ lies on a circle of radius A. At time $t = 0$, $\mathbf{f} = A\cos(kz)\,\hat{\mathbf{x}} - A\sin(kz)\,\hat{\mathbf{y}}$. At time $t = \pi/2\omega$, $\mathbf{f} = A\cos(kz - 90°)\,\hat{\mathbf{x}} - A\sin(kz - 90°)\,\hat{\mathbf{y}} = A\sin(kz)\,\hat{\mathbf{x}} + A\cos(kz)\,\hat{\mathbf{y}}$. Evidently it circles $\boxed{\text{counterclockwise}}$. To make a wave circling the other way, use $\delta_h = -90°$.



(b)

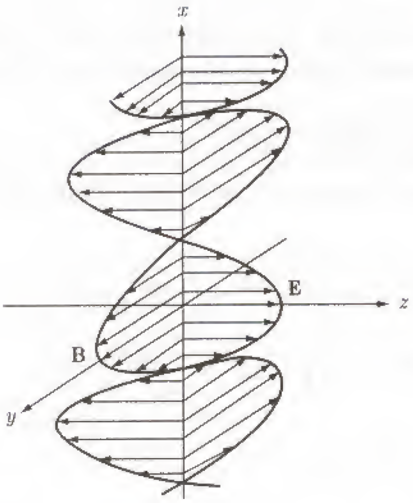


(c) Shake it around in a circle, instead of up and down.

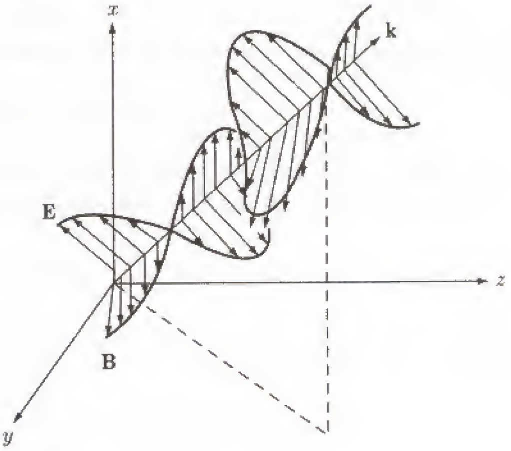
Problem 9.9

(a) $\boxed{\mathbf{k} = -\frac{\omega}{c}\,\hat{\mathbf{x}};\ \hat{\mathbf{n}} = \hat{\mathbf{z}}.}$ $\mathbf{k}\cdot\mathbf{r} = \left(-\frac{\omega}{c}\,\hat{\mathbf{x}}\right)\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = -\frac{\omega}{c}x;$ $\mathbf{k}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}.$

$$\boxed{\mathbf{E}(x,t) = E_0\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{z}};\quad \mathbf{B}(x,t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{y}}.}$$



(a)



(b)

(b) $\boxed{\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{3}}\right);\ \hat{\mathbf{n}} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}.}$ (Since $\hat{\mathbf{n}}$ is parallel to the $x\,z$ plane, it must have the form $\alpha\,\hat{\mathbf{x}} + \beta\,\hat{\mathbf{z}}$; since $\hat{\mathbf{n}}\cdot\mathbf{k} = 0, \beta = -\alpha$; and since it is a unit vector, $\alpha = 1/\sqrt{2}$.)

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x + y + z);\ \hat{\mathbf{k}}\times\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - \hat{\mathbf{z}}).$$

$$\boxed{\begin{aligned} \mathbf{E}(x,y,z,t) &= E_0\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left(\frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}\right); \\ \mathbf{B}(x,y,z,t) &= \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left(\frac{-\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}}}{\sqrt{6}}\right). \end{aligned}}$$

Problem 9.10

$P = \frac{I}{c} = \frac{1.3\times10^{3}}{3.0\times10^{8}} = \boxed{4.3\times10^{-6}\,\mathrm{N/m^2}.}$ For a perfect reflector the pressure is twice as great: $\boxed{8.6\times10^{-6}\,\mathrm{N/m^2}.}$ Atmospheric pressure is $1.03\times10^{5}\,\mathrm{N/m^2}$, so the pressure of light on a reflector is $(8.6\times10^{-6})/(1.03\times10^{5}) = \boxed{8.3\times10^{-11}\text{ atmospheres.}}$

Problem 9.11

$$\begin{aligned}\langle fg\rangle &= \frac{1}{T}\int_0^T a\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_a)b\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta_b)\,dt\\ &= \frac{ab}{2T}\int_0^T[\cos(2\mathbf{k}\cdot\mathbf{r}-2\omega t+\delta_a+\delta_b)+\cos(\delta_a-\delta_b)]\,dt=\frac{ab}{2T}\cos(\delta_a-\delta_b)T=\frac{1}{2}ab\cos(\delta_a-\delta_b).\end{aligned}$$

Meanwhile, in the complex notation: $\tilde{f}=\tilde{a}e^{i\mathbf{k}\cdot\mathbf{r}-\omega t)}$, $\tilde{g}=\tilde{b}e^{i\mathbf{k}\cdot\mathbf{r}-\omega t)}$, where $\tilde{a}=ae^{i\delta_a}$, $\tilde{b}=be^{i\delta_b}$. So $\frac{1}{2}\tilde{f}\tilde{g}^*=\frac{1}{2}\tilde{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\tilde{b}^*e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=\frac{1}{2}\tilde{a}\tilde{b}^*=\frac{1}{2}abe^{i(\delta_a-\delta_b)}$, $\quad\text{Re}\left(\frac{1}{2}\tilde{f}\tilde{g}^*\right)=\frac{1}{2}ab\cos(\delta_a-\delta_b)=\langle fg\rangle$. qed

Problem 9.12

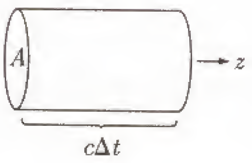
$$T_{ij}=\epsilon_0\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right).$$

With the fields in Eq. 9.48, **E** has only an x component, and **B** only a y component. So all the "off-diagonal" ($i\neq j$) terms are zero. As for the "diagonal" elements:

$$\begin{aligned}T_{xx} &= \epsilon_0\left(E_xE_x-\frac{1}{2}E^2\right)+\frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right)=\frac{1}{2}\left(\epsilon_0E^2-\frac{1}{\mu_0}B^2\right)=0.\\ T_{yy} &= \epsilon_0\left(-\frac{1}{2}E^2\right)+\frac{1}{\mu_0}\left(B_yB_y-\frac{1}{2}B^2\right)=\frac{1}{2}\left(-\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)=0.\\ T_{zz} &= \epsilon_0\left(-\frac{1}{2}E^2\right)+\frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right)=-u.\end{aligned}$$

So $\boxed{T_{zz}=-\epsilon_0E_0^2\cos^2(kz-\omega t+\delta)}$ (all other elements zero).

The momentum of these fields is in the z direction, and it is being *transported* in the z direction, so *yes*, it does make sense that T_{zz} should be the only nonzero element in T_{ij}. According to Sect. 8.2.3, $-\overleftrightarrow{T}\cdot d\mathbf{a}$ is the rate at which momentum crosses an area $d\mathbf{a}$. Here we have *no* momentum crossing areas oriented in the x or y direction; the momentum per unit time per unit area flowing across a surface oriented in the z direction is $-T_{zz}=u=\wp c$ (Eq. 9.59), so $\Delta p=\wp cA\Delta t$, and hence $\Delta p/\Delta t=\wp cA=$ momentum per unit time crossing area A. Evidently $\boxed{\text{momentum flux density = energy density.}}$ ✓



Problem 9.13

$R=\left(\frac{E_{0_R}}{E_{0_I}}\right)^2$ (Eq. 9.86) $\Rightarrow\boxed{R=\left(\frac{1-\beta}{1+\beta}\right)^2}$ (Eq. 9.82), where $\beta\equiv\frac{\mu_1v_1}{\mu_2v_2}$. $\quad T=\frac{\epsilon_2v_2}{\epsilon_1v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2$ (Eq. 9.87)

$\Rightarrow\boxed{T=\beta\left(\frac{2}{1+\beta}\right)^2}$ (Eq. 9.82). [Note that $\frac{\epsilon_2v_2}{\epsilon_1v_1}=\frac{\mu_1}{\mu_2}\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}\frac{v_2}{v_1}=\frac{\mu_1}{\mu_2}\left(\frac{v_1}{v_2}\right)^2\frac{v_2}{v_1}=\frac{\mu_1v_1}{\mu_2v_2}=\beta$.]

$$T+R=\frac{1}{(1+\beta)^2}\left[4\beta+(1-\beta)^2\right]=\frac{1}{(1+\beta)^2}(4\beta+1-2\beta+\beta^2)=\frac{1}{(1+\beta)^2}(1+2\beta+\beta^2)=1.\ ✓$$

Problem 9.14

Equation 9.78 is replaced by $\tilde{E}_{0_I}\hat{\mathbf{x}} + \tilde{E}_{0_R}\hat{\mathbf{n}}_R = \tilde{E}_{0_T}\hat{\mathbf{n}}_T$, and Eq. 9.80 becomes $\tilde{E}_{0_I}\hat{\mathbf{y}} - \tilde{E}_{0_R}(\hat{\mathbf{z}} \times \hat{\mathbf{n}}_R) = \beta\tilde{E}_{0_T}(\hat{\mathbf{z}} \times \hat{\mathbf{n}}_T)$. The y component of the first equation is $\tilde{E}_{0_R}\sin\theta_R = \tilde{E}_{0_T}\sin\theta_T$; the x component of the second is $\tilde{E}_{0_R}\sin\theta_R = -\beta\tilde{E}_{0_T}\sin\theta_T$. Comparing these two, we conclude that $\sin\theta_R = \sin\theta_T = 0$, and hence $\theta_R = \theta_T = 0$. qed

Problem 9.15

$Ae^{iax} + Be^{ibx} = Ce^{icx}$ for all x, so (using $x = 0$), $A + B = C$.

Differentiate: $iaAe^{iax} + ibBe^{ibx} = icCe^{icx}$, so (using $x = 0$), $aA + bB = cC$.

Differentiate again: $-a^2Ae^{iax} - b^2Be^{ibx} = -c^2Ce^{icx}$, so (using $x = 0$), $a^2A + b^2B = c^2C$.

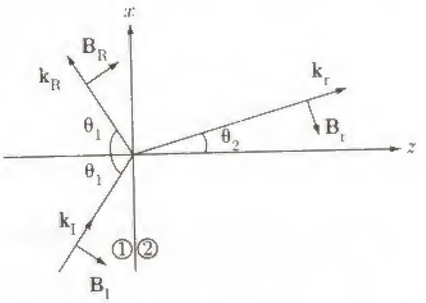
$a^2A + b^2B = c(cC) = c(aA + bB)$; $(A + B)(a^2A + b^2B) = (A + B)c(aA + bB) = cC(aA + bB)$;
$a^2A^2 + b^2AB + a^2AB + b^2B^2 = (aA + bB)^2 = a^2A^2 + 2abAB + b^2B^2$, or $(a^2 + b^2 - 2ab)AB = 0$, or $(a - b)^2AB = 0$. But A and B are nonzero, so $a = b$. Therefore $(A + B)e^{iax} = Ce^{icx}$.
$a(A + B) = cC$, or $aC = cC$, so (since $C \neq 0$) $a = c$. *Conclusion:* $a = b = c$. qed

Problem 9.16

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_I & = & \tilde{E}_{0_I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_I & = & \frac{1}{v_1}\tilde{E}_{0_I}e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}(-\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_R & = & \tilde{E}_{0_R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_R & = & \frac{1}{v_1}\tilde{E}_{0_R}e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}(\cos\theta_1\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$

$$\left\{\begin{array}{lcl} \tilde{\mathbf{E}}_T & = & \tilde{E}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}}, \\ \tilde{\mathbf{B}}_T & = & \frac{1}{v_2}\tilde{E}_{0_T}e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_2\,\hat{\mathbf{x}} + \sin\theta_1\,\hat{\mathbf{z}}); \end{array}\right\}$$



Boundary conditions: $\left\{\begin{array}{ll} \text{(i) } \epsilon_1E_1^{\perp} = \epsilon_2E_2^{\perp}, & \text{(iii) } \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \\ \text{(ii) } B_1^{\perp} = B_2^{\perp}, & \text{(iv) } \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}. \end{array}\right.$

Law of refraction: $\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{v_2}{v_1}$. [Note: $\mathbf{k}_I\cdot\mathbf{r} - \omega t = \mathbf{k}_R\cdot\mathbf{r} - \omega t = \mathbf{k}_T\cdot\mathbf{r} - \omega t$, at $z = 0$, so we can drop all exponential factors in applying the boundary conditions.]

Boundary condition (i): $0 = 0$ (trivial). Boundary condition (iii): $\boxed{\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}.}$

Boundary condition (ii): $\dfrac{1}{v_1}\tilde{E}_{0_I}\sin\theta_1 + \dfrac{1}{v_1}\tilde{E}_{0_R}\sin\theta_1 = \dfrac{1}{v_2}\tilde{E}_{0_T}\sin\theta_2 \Rightarrow \tilde{E}_{0_I} + \tilde{E}_{0_R} = \left(\dfrac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_{0_T}$.
But the term in parentheses is 1, by the law of refraction, so this is the same as (ii).

Boundary condition (iv): $\dfrac{1}{\mu_1}\left[\dfrac{1}{v_1}\tilde{E}_{0_I}(-\cos\theta_1) + \dfrac{1}{v_1}\tilde{E}_{0_R}\cos\theta_1\right] = \dfrac{1}{\mu_2v_2}\tilde{E}_{0_T}(-\cos\theta_2) \Rightarrow$

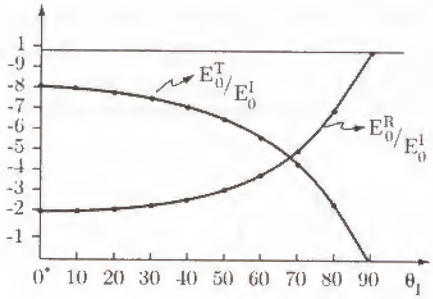
$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \left(\dfrac{\mu_1v_1\cos\theta_2}{\mu_2v_2\cos\theta_1}\right)\tilde{E}_{0_T}$. Let $\boxed{\alpha \equiv \dfrac{\cos\theta_2}{\cos\theta_1};\ \beta \equiv \dfrac{\mu_1v_1}{\mu_2v_2}.}$ Then $\boxed{\tilde{E}_{0_I} - \tilde{E}_{0_R} = \alpha\beta\tilde{E}_{0_T}.}$

Solving for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$: $2\tilde{E}_{0_I} = (1 + \alpha\beta)\tilde{E}_{0_T} \Rightarrow \tilde{E}_{0_T} = \left(\dfrac{2}{1 + \alpha\beta}\right)\tilde{E}_{0_I}$;

$\tilde{E}_{0_R} = \tilde{E}_{0_T} - \tilde{E}_{0_I} = \left(\dfrac{2}{1 + \alpha\beta} - \dfrac{1 + \alpha\beta}{1 + \alpha\beta}\right)\tilde{E}_{0_I} \Rightarrow \tilde{E}_{0_R} = \left(\dfrac{1 - \alpha\beta}{1 + \alpha\beta}\right)\tilde{E}_{0_I}$.

Since α and β are positive, it follows that $2/(1 + \alpha\beta)$ is positive, and hence the *transmitted* wave is *in phase* with the incident wave, and the (real) amplitudes are related by $\boxed{E_{0_T} = \left(\dfrac{2}{1 + \alpha\beta}\right)E_{0_I}.}$ The *reflected* wave is in phase if $\alpha\beta < 1$ and 180° out of phase if $\alpha\beta < 1$; the (real) amplitudes are related by $\boxed{E_{0_R} = \left|\dfrac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}.}$ These are the **Fresnel equations** for polarization perpendicular to the plane of incidence.

To construct the graphs, note that $\alpha\beta = \beta\dfrac{\sqrt{1-\sin^2\theta/\beta^2}}{\cos\theta} = \dfrac{\sqrt{\beta^2-\sin^2\theta}}{\cos\theta}$, where θ is the angle of incidence, so, for $\beta = 1.5$, $\alpha\beta = \dfrac{\sqrt{2.25-\sin^2\theta}}{\cos\theta}$.



Is there a Brewster's angle? Well, $E_{0_R} = 0$ would mean that $\alpha\beta = 1$, and hence that $\alpha = \dfrac{\sqrt{1-(v_2/v_1)^2\sin^2\theta}}{\cos\theta} = \dfrac{1}{\beta} = \dfrac{\mu_2 v_2}{\mu_1 v_1}$, or $1 - \left(\dfrac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\dfrac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta$, so $1 = \left(\dfrac{v_2}{v_1}\right)^2 \left[\sin^2\theta + (\mu_2/\mu_1)^2\cos^2\theta\right]$. Since $\mu_1 \approx \mu_2$, this means $1 \approx (v_2/v_1)^2$, which is only true for optically indistinguishable media, in which case there is of *course* no reflection—but that would be true at *any* angle, not just at a special "Brewster's angle". [If μ_2 were substantially different from μ_1, and the relative velocities were just right, it *would* be possible to get a Brewster's angle for this case, at

$$\left(\frac{v_1}{v_2}\right)^2 = 1 - \cos^2\theta + \left(\frac{\mu_2}{\mu_1}\right)^2 \cos^2\theta \Rightarrow \cos^2\theta = \frac{(v_1/v_2)^2 - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\mu_2\epsilon_2/\mu_1\epsilon_1) - 1}{(\mu_2/\mu_1)^2 - 1} = \frac{(\epsilon_2/\epsilon_1) - (\mu_1/\mu_2)}{(\mu_2/\mu_1) - (\mu_1/\mu_2)}.$$

But the media would be very peculiar.]

By the same token, δ_R is either always 0, or always π, for a given interface—it does not switch over as you change θ, the way it does for polarization in the plane of incidence. In particular, if $\beta = 3/2$, then $\alpha\beta > 1$, for

$$\alpha\beta = \frac{\sqrt{2.25-\sin^2\theta}}{\cos\theta} > 1 \text{ if } 2.25 - \sin^2\theta > \cos^2\theta, \text{ or } 2.25 > \sin^2\theta + \cos^2\theta = 1. \checkmark$$

In general, for $\beta > 1$, $\alpha\beta > 1$, and hence $\delta_R = \pi$. For $\beta < 1$, $\alpha\beta < 1$, and $\delta_R = 0$.

At normal incidence, $\alpha = 1$, so Fresnel's equations reduce to $E_{0_T} = \left(\dfrac{2}{1+\beta}\right) E_{0_I}$; $E_{0_R} = \left|\dfrac{1-\beta}{1+\beta}\right| E_{0_I}$, consistent with Eq. 9.82.

Reflection and Transmission coefficients: $\boxed{R = \left(\dfrac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2.}$ Referring to Eq. 9.116,

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \boxed{\alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2.}$$

$$R + T = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1. \checkmark$$

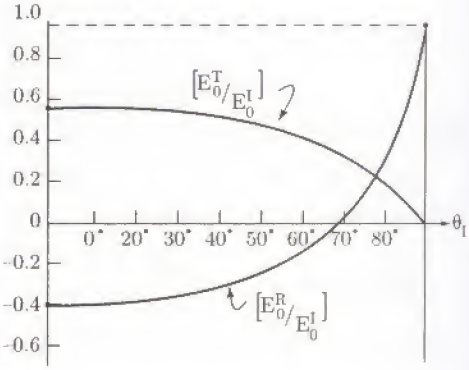
Problem 9.17

Equation 9.106 ⇒ $\beta = 2.42$; Eq. 9.110 ⇒
$\alpha = \dfrac{\sqrt{1-(\sin\theta/2.42)^2}}{\cos\theta}$.

(a) $\theta = 0 \Rightarrow \alpha = 1$. Eq. 9.109 ⇒ $\left(\dfrac{E_{0_R}}{E_{0_I}}\right) = \dfrac{\alpha-\beta}{\alpha+\beta} = \dfrac{1-2.42}{1+2.42} = -\dfrac{1.42}{3.42} = \boxed{-0.415;}$
$\left(\dfrac{E_{0_T}}{E_{0_I}}\right) = \dfrac{2}{\alpha+\beta} = \dfrac{2}{3.42} = \boxed{0.585.}$

(b) Equation 9.112 ⇒ $\theta_B = \tan^{-1}(2.42) = \boxed{67.5^\circ.}$

(c) $E_{0_R} = E_{0_T} \Rightarrow \alpha - \beta = 2; \alpha = \beta + 2 = 4.42;$
$(4.42)^2\cos^2\theta = 1 - \sin^2\theta/(2.42)^2;$
$(4.42)^2(1-\sin^2\theta) = (4.42)^2 - (4.42)^2\sin^2\theta$
$= 1 - 0.171\sin^2\theta;\ 19.5 - 1 = (19.5 - 0.17)\sin^2\theta;$
$18.5 = 19.3\sin^2\theta;\ \sin^2\theta = 18.5/19.3 = 0.959;$
$\sin\theta = 0.979;\ \boxed{\theta = 78.3^\circ.}$



Problem 9.18

(a) Equation 9.120 ⇒ $\tau = \epsilon/\sigma$. Now $\epsilon = \epsilon_0\epsilon_r$ (Eq. 4.34), $\epsilon_r \cong n^2$ (Eq. 9.70), and for glass the index of refraction is typically around 1.5, so $\epsilon \approx (1.5)^2 \times 8.85 \times 10^{-12} = 2 \times 10^{-11}\ \mathrm{C^2/N\,m^2}$, while $\sigma = 1/\rho \approx 10^{-12}\ \Omega\,\mathrm{m}$ (Table 7.1). Then $\tau = (2\times 10^{-11})/10^{-12} = \boxed{20\ \mathrm{s.}}$ (But the resistivity of glass varies enormously from one type to another, so this answer could be off by a factor of 100 in either direction.)

(b) For silver, $\rho = 1.59 \times 10^{-8}$ (Table 7.1), and $\epsilon \approx \epsilon_0$, so $\omega\epsilon = 2\pi \times 10^{10} \times 8.85 \times 10^{-12} = 0.56$. Since $\sigma = 1/\rho = 6.25 \times 10^7 \gg \omega\epsilon$, the skin depth (Eq. 9.128) is

$$d = \frac{1}{\kappa} \cong \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{2\pi\times 10^{10}\times 6.25\times 10^7 \times 4\pi\times 10^{-7}}} = 6.4\times 10^{-7}\ \mathrm{m} = 6.4\times 10^{-4}\ \mathrm{mm}.$$

I'd plate silver to a depth of about $\boxed{0.001\,\mathrm{mm};}$ there's no point in making it any thicker, since the fields don't penetrate much beyond this anyway.

(c) For copper, Table 7.1 gives $\sigma = 1/(1.68\times 10^{-8}) = 6\times 10^7$, $\omega\epsilon_0 = (2\pi\times 10^6)\times(8.85\times 10^{-12}) = 6\times 10^{-5}$. Since $\sigma \gg \omega\epsilon$, Eq. 9.126 ⇒ $k \approx \sqrt{\dfrac{\omega\sigma\mu}{2}}$, so (Eq. 9.129)

$$\lambda = 2\pi\sqrt{\frac{2}{\omega\sigma\mu_0}} = 2\pi\sqrt{\frac{2}{2\pi\times 10^6\times 6\times 10^7\times 4\pi\times 10^{-7}}} = 4\times 10^{-4}\ \mathrm{m} = \boxed{0.4\,\mathrm{mm.}}$$

¿From Eq. 9.129, the propagation speed is $v = \dfrac{\omega}{k} = \dfrac{\omega}{2\pi}\lambda = \lambda\nu = (4\times 10^{-4})\times 10^6 = \boxed{400\,\mathrm{m/s.}}$ In vacuum, $\lambda = \dfrac{c}{\nu} = \dfrac{3\times 10^8}{10^6} = \boxed{300\,\mathrm{m};}$ $v = c = \boxed{3\times 10^8\,\mathrm{m/s.}}$ (But really, in a good conductor the skin depth is so small, compared to the wavelength, that the notions of "wavelength" and "propagation speed" lose their meaning.)

Problem 9.19

(a) Use the binomial expansion for the square root in Eq. 9.126:

$$\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2-1\right]^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}.$$

So (Eq. 9.128) $d = \dfrac{1}{\kappa} \cong \dfrac{2}{\sigma}\sqrt{\dfrac{\epsilon}{\mu}}$. qed

For pure water, $\left\{\begin{array}{l} \epsilon = \epsilon_r\epsilon_0 = 80.1\,\epsilon_0 \quad \text{(Table 4.2)}, \\ \mu = \mu_0(1+\chi_m) = \mu_0(1-9.0\times10^{-6}) \cong \mu_0 \quad \text{(Table 6.1)}, \\ \sigma = 1/(2.5\times10^5) \quad \text{(Table 7.1)}. \end{array}\right.$

So $d = (2)(2.5\times10^5)\sqrt{\dfrac{(80.1)(8.85\times10^{-12})}{4\pi\times10^{-7}}} = \boxed{1.19\times10^4\,\text{m.}}$

(b) In this case $(\sigma/\epsilon\omega)^2$ dominates, so (Eq. 9.126) $k \cong \kappa$, and hence (Eqs. 9.128 and 9.129) $\lambda = \dfrac{2\pi}{k} \cong \dfrac{2\pi}{\kappa} = 2\pi d$, or $d = \dfrac{\lambda}{2\pi}$. qed

Meanwhile $\kappa \cong \omega\sqrt{\dfrac{\epsilon\mu}{2}}\sqrt{\dfrac{\sigma}{\epsilon\omega}} = \sqrt{\dfrac{\omega\mu\sigma}{2}} = \sqrt{\dfrac{(10^{15})(4\pi\times10^{-7})(10^7)}{2}} = 8\times10^7$; $d = \dfrac{1}{\kappa} = \dfrac{1}{8\times10^7} = 1.3\times10^{-8} = \boxed{13\,\text{nm.}}$ So the fields do not penetrate far into a metal—which is what accounts for their opacity.

(c) Since $k \cong \kappa$, as we found in (b), Eq. 9.134 says $\phi = \tan^{-1}(1) = 45°$. qed

Meanwhile, Eq. 9.137 says $\dfrac{B_0}{E_0} \cong \sqrt{\epsilon\mu\dfrac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\dfrac{\sigma\mu}{\omega}}.}$ For a typical metal, then, $\dfrac{B_0}{E_0} = \sqrt{\dfrac{(10^7)(4\pi\times10^{-7})}{10^{15}}} = \boxed{10^{-7}\,\text{s/m.}}$ (In vacuum, the ratio is $1/c = 1/(3\times10^8) = 3\times10^{-9}$ s/m, so the magnetic field is comparatively about 100 times larger in a metal.)

Problem 9.20

(a) $u = \dfrac{1}{2}\left(\epsilon E^2 + \dfrac{1}{\mu}B^2\right) = \dfrac{1}{2}e^{-2\kappa z}\left[\epsilon E_0^2\cos^2(kz-\omega t+\delta_E) + \dfrac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\phi)\right]$. Averaging over a full cycle, using $\langle\cos^2\rangle = \frac{1}{2}$ and Eq. 9.137:

$$\langle u\rangle = \frac{1}{2}e^{-2\kappa z}\left[\frac{\epsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right] = \frac{1}{4}e^{-2\kappa z}\left[\epsilon E_0^2 + \frac{1}{\mu}E_0^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] = \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right].$$

But Eq. 9.126 $\Rightarrow 1+\sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2} = \dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2}$, so $\langle u\rangle = \dfrac{1}{4}e^{-2\kappa z}\epsilon E_0^2\dfrac{2}{\epsilon\mu}\dfrac{k^2}{\omega^2} = \boxed{\dfrac{k^2}{2\mu\omega^2}E_0^2e^{-2\kappa z}.}$ So the ratio of the magnetic contribution to the electric contribution is

$$\frac{\langle u_{\text{mag}}\rangle}{\langle u_{\text{elec}}\rangle} = \frac{B_0^2/\mu}{E_0^2\epsilon} = \frac{1}{\mu\epsilon}\mu\epsilon\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} = \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} > 1. \quad \text{qed}$$

(b) $\mathbf{S} = \dfrac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = \dfrac{1}{\mu}E_0B_0e^{-2\kappa z}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{z}}$; $\langle\mathbf{S}\rangle = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi\,\hat{\mathbf{z}}$. [The average of the product of the cosines is $(1/2\pi)\int_0^{2\pi}\cos\theta\cos(\theta+\phi)\,d\theta = (1/2)\cos\phi$.] So $I = \dfrac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi = \dfrac{1}{2\mu}E_0^2e^{-2\kappa z}\left(\dfrac{K}{\omega}\cos\phi\right)$, while, from Eqs. 9.133 and 9.134, $K\cos\phi = k$, so $\boxed{I = \dfrac{k}{2\mu\omega}E_0^2e^{-2\kappa z}.}$ qed

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Problem 9.21

According to Eq. 9.147, $R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right)$, where $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$ $= \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i\kappa_2)$ (Eqs. 9.125 and 9.146). Since silver is a good conductor ($\sigma \gg \epsilon\omega$), Eq. 9.126 reduces to $\kappa_2 \cong k_2 \cong \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\sqrt{\frac{\sigma}{\epsilon_2\omega}} = \sqrt{\frac{\sigma\omega\mu_2}{2}}$, so $\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i) = \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}}(1+i)$.

Let $\gamma \equiv \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}} = \mu_0 c\sqrt{\frac{\sigma}{2\mu_0\omega}} = c\sqrt{\frac{\sigma\mu_0}{2\omega}} = (3\times10^8)\sqrt{\frac{(6\times10^7)(4\pi\times10^{-7})}{(2)(4\times10^{15})}} = 29.$ Then

$R = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} = \boxed{0.93.}$ Evidently 93% of the light is reflected.

Problem 9.22

(a) We are told that $v = \alpha\sqrt{\lambda}$, where α is a constant. But $\lambda = 2\pi/k$ and $v = \omega/k$, so $\omega = \alpha k\sqrt{2\pi/k} = \alpha\sqrt{2\pi k}$. From Eq. 9.150, $v_g = \frac{d\omega}{dk} = \alpha\sqrt{2\pi}\frac{1}{2\sqrt{k}} = \frac{1}{2}\alpha\sqrt{\frac{2\pi}{k}} = \frac{1}{2}\alpha\sqrt{\lambda} = \frac{1}{2}v$, or $\boxed{v = 2v_g.}$

(b) $\frac{i(px-Et)}{\hbar} = i(kx-\omega t) \Rightarrow k = \frac{p}{\hbar},\ \omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar} = \frac{\hbar k^2}{2m}$. Therefore $\boxed{v = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \frac{\hbar k}{2m};}$ $v_g = \frac{d\omega}{dk} = \frac{2\hbar k}{2m} = \frac{\hbar k}{m} = \boxed{\frac{p}{m}.}$ So $\boxed{v = \frac{1}{2}v_g.}$ Since $p = mv_c$ (where v_c is the classical speed of the particle), it follows that $\boxed{v_g \text{ (not } v\text{) corresponds to the classical veloctity.}}$

Problem 9.23

$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} \Rightarrow F = -qE = -\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^3}\right)x = -k_{\text{spring}}x = -m\omega_0^2 x$ (Eq. 9.151). So $\boxed{\omega_0 = \sqrt{\frac{q^2}{4\pi\epsilon_0 m a^3}}.}$

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(0.5\times10^{-10})^3}} = \boxed{7.16\times10^{15}\text{ Hz.}}$$ This is $\boxed{\text{ultraviolet.}}$

¿From Eqs. 9.173 and 9.174,

$$A = \frac{nq^2}{2m\epsilon_0}\frac{f}{\omega_0^2},\ \left\{\begin{array}{lcl} N & = & \#\text{ of molecules per unit volume } = \frac{\text{Avogadro's \#}}{22.4\text{ liters}} = \frac{6.02\times10^{23}}{22.4\times10^{-3}} = 2.69\times10^{25}, \\ f & = & \#\text{ of electrons per molecule } = 2\text{ (for H}_2\text{)}. \end{array}\right.$$

$$= \frac{(2.69\times10^{25})(1.6\times10^{-19})^2}{(9.11\times10^{-31})(8.85\times10^{-12})(4.5\times10^{16})^2} = \boxed{4.2\times10^{-5}}$$ (which is about 1/3 the actual value);

$$B = \left(\frac{2\pi c}{\omega_0}\right)^2 = \left(\frac{2\pi\times3\times10^8}{4.5\times10^{16}}\right)^2 = \boxed{1.8\times10^{-15}\text{ m}^2}$$ (which is about 1/4 the actual value).

So even this extremely crude model is in the right ball park.

Problem 9.24

Equation 9.170 $\Rightarrow n = 1 + \frac{Nq^2}{2m\epsilon_0}\frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$. Let the denominator $\equiv D$. Then

$$\frac{dn}{d\omega} = \frac{Nq^2}{2m\epsilon_0}\left\{\frac{-2\omega}{D} - \frac{(\omega_0^2-\omega^2)}{D^2}\left[2(\omega_0^2-\omega^2)(-2\omega)+\gamma^2 2\omega\right]\right\} = 0 \Rightarrow 2\omega D = (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)-\gamma^2\right]2\omega;$$

$(\omega_0^2-\omega^2)^2+\gamma^2\omega^2 = 2(\omega_0^2-\omega^2)^2-\gamma^2(\omega_0^2-\omega^2)$, or $(\omega_0^2-\omega^2)^2 = \gamma^2(\omega^2+\omega_0^2-\omega^2) = \gamma^2\omega_0^2 \Rightarrow (\omega_0^2-\omega^2) = \pm\omega_0\gamma;$

$\omega^2 = \omega_0^2 \mp \omega_0\gamma$, $\omega = \omega_0\sqrt{1 \mp \gamma/\omega_0} \cong \omega_0\left(1 \mp \gamma/2\omega_0\right) = \omega_0 \mp \gamma/2$. So $\omega_2 = \omega_0 + \gamma/2$, $\omega_1 = \omega_0 - \gamma/2$, and the width of the anomalous region is $\boxed{\Delta\omega = \omega_2 - \omega_1 = \gamma.}$

From Eq. 9.171, $\alpha = \dfrac{Nq^2\omega^2}{m\epsilon_0 c}\dfrac{\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$, so at the maximum ($\omega = \omega_0$), $\alpha_{\text{max}} = \dfrac{Nq^2}{m\epsilon_0 c\gamma}$.

At ω_1 and ω_2, $\omega^2 = \omega_0^2 \mp \omega_0\gamma$, so $\alpha = \dfrac{Nq^2\omega^2}{m\epsilon_0 c}\dfrac{\gamma}{\gamma^2\omega_0^2 + \gamma^2\omega^2} = \alpha_{\text{max}}\left(\dfrac{\omega^2}{\omega^2 + \omega_0^2}\right)$. But

$$\frac{\omega^2}{\omega^2 + \omega_0^2} = \frac{\omega_0^2 \mp \omega_0\gamma}{2\omega_0^2 \mp \omega_0\gamma} = \frac{1}{2}\frac{(1 \mp \gamma/\omega_0)}{(1 \mp \gamma/2\omega_0)} \cong \frac{1}{2}\left(1 \mp \frac{\gamma}{\omega_0}\right)\left(1 \pm \frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}\left(1 \mp \frac{\gamma}{2\omega_0}\right) \cong \frac{1}{2}.$$

So $\alpha \cong \frac{1}{2}\alpha_{\text{max}}$ at ω_1 and ω_2. qed

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Problem 9.25

$$k = \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2 - \omega^2)}\right]. \quad v_g = \frac{d\omega}{dk} = \frac{1}{(dk/d\omega)}.$$

$$\frac{dk}{d\omega} = \frac{1}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j}{(\omega_j^2 - \omega^2)} + \omega\sum f_j\frac{-(-2\omega)}{(\omega_j^2 - \omega^2)^2}\right] = \frac{1}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum f_j\frac{(\omega_j^2 + \omega^2)}{(\omega_j^2 - \omega^2)^2}\right].$$

$\boxed{v_g = c\left[1 + \dfrac{Nq^2}{2m\epsilon_0}\sum f_j\dfrac{(\omega_j^2 + \omega^2)}{(\omega_j^2 - \omega^2)^2}\right]^{-1}.}$ Since the second term in square brackets is *positive*, it follows that $\boxed{v_g < c,}$ whereas $v = \dfrac{\omega}{k} = c\left[1 + \dfrac{Nq^2}{2m\epsilon_0}\sum\dfrac{f_j}{(\omega_j^2 - \omega^2)}\right]^{-1}$ is greater than c or less than c, depending on ω.

Problem 9.26

(a) From Eqs. 9.176 and 9.177, $\nabla \times \tilde{\mathbf{E}} = -\dfrac{\partial\tilde{\mathbf{B}}}{\partial t} = i\omega\tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$; $\nabla \times \tilde{\mathbf{B}} = \dfrac{1}{c^2}\dfrac{\partial\tilde{\mathbf{E}}}{\partial t} = -\dfrac{i\omega}{c^2}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$.
In the terminology of Eq. 9.178:

$$(\nabla \times \tilde{\mathbf{E}})_x = \frac{\partial\tilde{E}_z}{\partial y} - \frac{\partial\tilde{E}_y}{\partial z} = \left(\frac{\partial\tilde{E}_{0_z}}{\partial y} - ik\tilde{E}_{0_y}\right)e^{i(kz-\omega t)}. \quad \text{So (ii) } \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x.$$

$$(\nabla \times \tilde{\mathbf{E}})_y = \frac{\partial\tilde{E}_x}{\partial z} - \frac{\partial\tilde{E}_z}{\partial x} = \left(ik\tilde{E}_{0_x} - \frac{\partial\tilde{E}_{0_z}}{\partial x}\right)e^{i(kz-\omega t)}. \quad \text{So (iii) } ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y.$$

$$(\nabla \times \tilde{\mathbf{E}})_z = \frac{\partial\tilde{E}_y}{\partial x} - \frac{\partial\tilde{E}_x}{\partial y} = \left(\frac{\partial\tilde{E}_{0_y}}{\partial x} - \frac{\partial\tilde{E}_{0_x}}{\partial y}\right)e^{i(kz-\omega t)}. \quad \text{So (i) } \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z.$$

$$(\nabla \times \tilde{\mathbf{B}})_x = \frac{\partial\tilde{B}_z}{\partial y} - \frac{\partial\tilde{B}_y}{\partial z} = \left(\frac{\partial\tilde{B}_{0_z}}{\partial y} - ik\tilde{B}_{0_y}\right)e^{i(kz-\omega t)}. \quad \text{So (v) } \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x.$$

$$(\nabla \times \tilde{\mathbf{B}})_y = \frac{\partial\tilde{B}_x}{\partial z} - \frac{\partial\tilde{B}_z}{\partial x} = \left(ik\tilde{B}_{0_x} - \frac{\partial\tilde{B}_{0_z}}{\partial x}\right)e^{i(kz-\omega t)}. \quad \text{So (vi) } ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y.$$

$$(\nabla \times \tilde{\mathbf{B}})_z = \frac{\partial\tilde{B}_y}{\partial x} - \frac{\partial\tilde{B}_x}{\partial y} = \left(\frac{\partial\tilde{B}_{0_y}}{\partial x} - \frac{\partial\tilde{B}_{0_x}}{\partial y}\right)e^{i(kz-\omega t)}. \quad \text{So (iv) } \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z.$$

This confirms Eq. 9.179. Now multiply (iii) by k, (v) by ω, and subtract: $ik^2E_x - k\dfrac{\partial E_z}{\partial x} - \omega\dfrac{\partial B_z}{\partial y} + i\omega kB_y = ik\omega B_y + \dfrac{i\omega^2}{c^2}E_x \Rightarrow i\left(k^2 - \dfrac{\omega^2}{c^2}\right)E_x = k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}$, or (i) $E_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right)$.

Multiply (ii) by k, (vi) by ω, and add: $k\dfrac{\partial E_z}{\partial y} - ik^2E_y + i\omega kB_x - \omega\dfrac{\partial B_z}{\partial x} = i\omega kB_x - \dfrac{i\omega^2}{c^2}E_y \Rightarrow i\left(\dfrac{\omega^2}{c^2} - k^2\right)E_y =$

$-k\frac{\partial E_z}{\partial y} + \omega\frac{\partial B_z}{\partial x}$, or (ii) $E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$.

Multiply (ii) by ω/c^2, (vi) by k, and add: $\frac{\omega}{c^2}\frac{\partial E_z}{\partial y} - i\frac{\omega k}{c^2}E_y + ik^2B_x - k\frac{\partial B_z}{\partial x} = i\frac{\omega^2}{c^2}B_x - i\frac{\omega k}{c^2}E_y \Rightarrow$ $i\left(k^2 - \frac{\omega^2}{c^2}\right)B_x = k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}$, or (iii) $B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$.

Multiply (iii) by ω/c^2, (v) by k, and subtract: $i\frac{\omega k}{c^2}E_x - \frac{\omega}{c^2}\frac{\partial E_z}{\partial x} - k\frac{\partial B_z}{\partial y} + ik^2B_y = i\frac{\omega^2}{c^2}B_y + \frac{i\omega k}{c^2}E_x \Rightarrow$ $i\left(k^2 - \frac{\omega^2}{c^2}\right)B_y = \frac{\omega}{c^2}\frac{\partial E_z}{\partial x} + k\frac{\partial B_z}{\partial y}$, or (iv) $B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$.

This completes the confirmation of Eq. 9.180.

(b) $\boldsymbol{\nabla}\cdot\tilde{\mathbf{E}} = \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y} + \frac{\partial \tilde{E}_z}{\partial z} = \left(\frac{\partial \tilde{E}_{0_x}}{\partial x} + \frac{\partial \tilde{E}_{0_y}}{\partial y} + ik\tilde{E}_{0_z}\right)e^{i(kz-\omega t)} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0.$

Using Eq. 9.180, $\frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 E_z}{\partial x^2} + \omega\frac{\partial^2 B_z}{\partial x\partial y}\right) + \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 E_z}{\partial^2 y} - \omega\frac{\partial^2 B_z}{\partial x\partial y}\right) + ikE_z = 0,$

or $\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial^2 y} + \left[(\omega/c)^2 - k^2\right]E_z = 0.$

Likewise, $\boldsymbol{\nabla}\cdot\tilde{\mathbf{B}} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0 \Rightarrow$

$\frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 B_z}{\partial x^2} - \frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y}\right) + \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial^2 B_z}{\partial y^2} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x\partial y}\right) + ikB_z = 0 \Rightarrow$

$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial^2 y} + \left[(\omega/c)^2 - k^2\right]B_z = 0.$

This confirms Eqs. 9.181. [You can also do it by putting Eq. 9.180 into Eq. 9.179 (i) and (iv).]

Problem 9.27

Here $E_z = 0$ (TE) and $\omega/c = k$ $(n = m = 0)$, so Eq. 9.179(ii) $\Rightarrow E_y = -cB_x$, Eq. 9.179(iii) $\Rightarrow E_x = cB_y$, Eq. 9.179(v) $\Rightarrow \frac{\partial B_z}{\partial y} = i\left(kB_y - \frac{\omega}{c^2}E_x\right) = i\left(kB_y - \frac{\omega}{c}B_y\right) = 0$, Eq. 9.179(vi) $\Rightarrow \frac{\partial B_z}{\partial x} = i\left(kB_x + \frac{\omega}{c^2}E_y\right) = i\left(kB_x - \frac{\omega}{c}B_x\right) = 0$. So $\frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial y} = 0$, and since B_z is a function only of x and y, this says B_z is in fact a *constant* (as Eq. 9.186 also suggests). Now Faraday's law (in integral form) says $\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a}$, and Eq. 9.176 $\Rightarrow \frac{\partial \mathbf{B}}{\partial t} = -i\omega\mathbf{B}$, so $\oint \mathbf{E}\cdot d\mathbf{l} = i\omega\int \mathbf{B}\cdot d\mathbf{a}$. Applied to a cross-section of the waveguide this gives $\oint \mathbf{E}\cdot d\mathbf{l} = i\omega e^{i(kz-\omega t)}\int B_z\, da = i\omega B_z e^{i(kz-\omega t)}(ab)$ (since B_z is constant, it comes outside the integral). But if the boundary is just inside the metal, where $\mathbf{E} = 0$, it follows that $\boxed{B_z = 0.}$ So this would be a TEM mode, which we already know cannot exist for this guide.

Problem 9.28

Here $a = 2.28\,\text{cm}$ and $b = 1.01\,\text{cm}$, so $\nu_{10} = \frac{1}{2\pi}\omega_{10} = \frac{c}{2a} = 0.66 \times 10^{10}\,\text{Hz}$; $\nu_{20} = 2\frac{c}{2a} = 1.32 \times 10^{10}\,\text{Hz}$; $\nu_{30} = 3\frac{c}{2a} = 1.97 \times 10^{10}\,\text{Hz}$; $\nu_{01} = \frac{c}{2b} = 1.49 \times 10^{10}\,\text{Hz}$; $\nu_{02} = 2\frac{c}{2b} = 2.97 \times 10^{10}\,\text{Hz}$; $\nu_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1.62 \times 10^{10}\,\text{Hz}$. Evidently just four modes occur: $\boxed{\text{10, 20, 01, and 11.}}$

To get only *one* mode you must drive the waveguide at a frequency between ν_{10} and ν_{20}: $\boxed{0.66 \times 10^{10} < \nu < 1.32 \times 10^{10}\,\text{Hz.}}$ $\lambda = \frac{c}{\nu}$, so $\lambda_{10} = 2a$; $\lambda_{20} = a$. $\boxed{2.28\,\text{cm} < \lambda < 4.56\,\text{cm.}}$

Problem 9.29

From Prob. 9.11, $\langle \mathbf{S}\rangle = \dfrac{1}{2\mu_0}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)$. Here (Eq. 9.176) $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$, $\tilde{\mathbf{B}}^* = \tilde{\mathbf{B}}_0^* e^{-i(kz-\omega t)}$, and, for the TE$_{mn}$ mode (Eqs. 9.180 and 9.186)

$$\begin{aligned}
B_x^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
B_y^* &= \frac{-ik}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
B_z^* &= B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_x &= \frac{i\omega}{(\omega/c)^2-k^2}\left(\frac{-n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);\\
E_y &= \frac{-i\omega}{(\omega/c)^2-k^2}\left(\frac{-m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right);\\
E_z &= 0.
\end{aligned}$$

So

$$\begin{aligned}
\langle\mathbf{S}\rangle &= \frac{1}{2\mu_0}\left\{\frac{i\pi\omega B_0^2}{(\omega/c)^2-k^2}\left(\frac{m}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\hat{\mathbf{x}}\right.\\
&\quad+\frac{i\pi\omega B_0^2}{(\omega/c)^2-k^2}\left(\frac{n}{b}\right)\cos^2\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{y}}\\
&\quad\left.+\frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\hat{\mathbf{z}}\right\}.\\
\int\langle\mathbf{S}\rangle\cdot d\mathbf{a} &= \boxed{\frac{1}{8\mu_0}\frac{\omega k\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}ab\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right].}
\end{aligned}$$

[In the last step I used $\int_0^a\sin^2(m\pi x/a)\,dx = \int_0^a\cos^2(m\pi x/a)\,dx = a/2$; $\int_0^b\sin^2(n\pi y/b)\,dy = \int_0^b\cos^2(n\pi y/b)\,dy = b/2$.]

Similarly,

$$\begin{aligned}
\langle u\rangle &= \frac{1}{4}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^*+\frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)\\
&= \frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\\
&\quad+\frac{1}{4\mu_0}\left\{B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right.\\
&\quad\left.+\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)+\left(\frac{m}{a}\right)^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right]\right\}.
\end{aligned}$$

$$\int\langle u\rangle\,da = \boxed{\frac{ab}{4}\left\{\frac{\epsilon_0}{4}\frac{\omega^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]+\frac{B_0^2}{4\mu_0}+\frac{1}{4\mu_0}\frac{k^2\pi^2B_0^2}{[(\omega/c)^2-k^2]^2}\left[\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2\right]\right\}.}$$

These results can be simplified, using Eq. 9.190 to write $\left[(\omega/c)^2 - k^2\right] = (\omega_{mn}/c)^2$, $\epsilon_0\mu_0 = 1/c^2$ to eliminate ϵ_0, and Eq. 9.188 to write $\left[(m/a)^2 + (n/b)^2\right] = (\omega_{mn}/\pi c)^2$:

$$\int \langle\mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\omega k a b c^2}{8\mu_0\omega_{mn}^2} B_0^2; \qquad \int \langle u\rangle\, da = \frac{\omega^2 ab}{8\mu_0\omega_{mn}^2} B_0^2.$$

Evidently

$$\frac{\text{energy per unit time}}{\text{energy per unit length}} = \frac{\int\langle\mathbf{S}\rangle \cdot d\mathbf{a}}{\int \langle u\rangle\, da} = \frac{kc^2}{\omega} = \frac{c}{\omega}\sqrt{\omega^2 - \omega_{mn}^2} = v_g \text{ (Eq. 9.192).} \quad \text{qed}$$

Problem 9.30

Following Sect. 9.5.2, the problem is to solve Eq. 9.181 with $E_z \neq 0, B_z = 0$, subject to the boundary conditions 9.175. Let $E_z(x,y) = X(x)Y(y)$; as before, we obtain $X(x) = A\sin(k_x x) + B\cos(k_x x)$. But the boundary condition requires $E_z = 0$ (and hence $X = 0$) when $x = 0$ and $x = a$, so $B = 0$ and $k_x = m\pi/a$. But this time $m = 1, 2, 3, \dots$, but *not* zero, since $m = 0$ would kill X entirely. The same goes for $Y(y)$. Thus $\boxed{E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$ with $n, m = 1, 2, 3, \dots$.

The rest is the same as for TE waves: $\boxed{\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}}$ is the cutoff frequency, the wave velocity is $v = c/\sqrt{1 - (\omega_{mn}/\omega)^2}$, and the group velocity is $v_g = c\sqrt{1 - (\omega_{mn}/\omega)^2}$. The lowest TM mode is 11, with cutoff frequency $\omega_{11} = c\pi\sqrt{(1/a)^2 + (1/b)^2}$. So the ratio of the lowest TM frequency to the lowest TE frequency is $\dfrac{c\pi\sqrt{(1/a)^2 + (1/b)^2}}{(c\pi/a)} = \boxed{\sqrt{1 + (a/b)^2}.}$

Problem 9.31

(a) $\nabla\cdot\mathbf{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) = 0\ \checkmark$; $\nabla\cdot\mathbf{B} = \frac{1}{s}\frac{\partial}{\partial\phi}(B_\phi) = 0\ \checkmark$; $\nabla\times\mathbf{E} = \frac{\partial E_s}{\partial z}\,\hat{\boldsymbol{\phi}} - \frac{1}{s}\frac{\partial E_s}{\partial\phi}\,\hat{\mathbf{z}} = -\frac{E_0 k\sin(kz-\omega t)}{s}\,\hat{\boldsymbol{\phi}} \overset{?}{=} -\frac{\partial\mathbf{B}}{\partial t} = -\frac{E_0\omega}{c}\frac{\sin(kz-\omega t)}{s}\,\hat{\boldsymbol{\phi}}\ \checkmark$ (since $k = \omega/c$); $\nabla\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\,\hat{\mathbf{z}} = \frac{E_0 k}{c}\frac{\sin(kz-\omega t)}{s}\,\hat{\mathbf{s}} \overset{?}{=} \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \frac{E_0\omega}{c^2}\frac{\sin(kz-\omega t)}{s}\,\hat{\mathbf{s}}\ \checkmark$. Boundary conditions: $E^{\|} = E_z = 0\ \checkmark$; $B^{\perp} = B_s = 0\ \checkmark$.

(b) To determine λ, use Gauss's law for a cylinder of radius s and length dz:

$\oint \mathbf{E}\cdot d\mathbf{a} = E_0\frac{\cos(kz-\omega t)}{s}(2\pi s)\,dz = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\lambda\,dz \Rightarrow \boxed{\lambda = 2\pi\epsilon_0 E_0\cos(kz-\omega t).}$

To determine I, use Ampére's law for a circle of radius s (note that the displacement current through this loop is zero, since $\mathbf{E}$ is in the $\hat{\mathbf{s}}$ direction): $\oint \mathbf{B}\cdot d\mathbf{l} = \frac{E_0}{c}\frac{\cos(kz-\omega t)}{s}(2\pi s) = \mu_0 I_{\text{enc}} \Rightarrow \boxed{I = \frac{2\pi E_0}{\mu_0 c}\cos(kz-\omega t).}$

The charge and current on the outer conductor are precisely the $\boxed{\text{opposite}}$ of these, since $\mathbf{E} = \mathbf{B} = 0$ *inside* the metal, and hence the *total* enclosed charge and current must be zero.

Problem 9.32

$\tilde{f}(z,0) = \int_{-\infty}^{\infty}\tilde{A}(k)e^{ikz}\,dk \Rightarrow \tilde{f}(z,0)^* = \int_{-\infty}^{\infty}\tilde{A}(k)^* e^{-ikz}\,dk$. Let $l \equiv -k$; then $\tilde{f}(z,0)^* = \int_{\infty}^{-\infty}\tilde{A}(-l)^* e^{ilz}(-dl) = \int_{-\infty}^{\infty}\tilde{A}(-l)^* e^{ilz}\,dl = \int_{-\infty}^{\infty}\tilde{A}(-k)^* e^{ikz}\,dk$ (renaming the dummy variable $l \to k$).

$f(z,0) = \text{Re}\left[\tilde{f}(z,0)\right] = \frac{1}{2}\left[\tilde{f}(z,0) + \tilde{f}(z,0)^*\right] = \int_{-\infty}^{\infty}\frac{1}{2}\left[\tilde{A}(k) + \tilde{A}(-k)^*\right]e^{ikz}\,dk$. Therefore

$\frac{1}{2}\left[\tilde{A}(k)+\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(z,0)e^{-ikz}\,dz.$

Meanwhile, $\dot{\tilde{f}}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)(-i\omega)e^{i(kz-\omega t)}\,dk \Rightarrow \dot{\tilde{f}}(z,0)=\int_{-\infty}^{\infty}[-i\omega\tilde{A}(k)]e^{ikz}\,dk.$
(Note that $\omega=|k|v$, here, so it does *not* come outside the integral.)

$$\begin{aligned}
\dot{\tilde{f}}(z,0)^* &= \int_{-\infty}^{\infty}[i\omega\tilde{A}(k)^*]e^{-ikz}\,dk=\int_{-\infty}^{\infty}[i|k|v\tilde{A}(k)^*]e^{-ikz}\,dk=\int_{\infty}^{-\infty}[i|l|v\tilde{A}(-l)^*]e^{ilz}(-dl)\\
&= \int_{-\infty}^{\infty}[i|k|v\tilde{A}(-k)^*]e^{ikz}\,dk=\int_{-\infty}^{\infty}[i\omega\tilde{A}(-k)^*]e^{ikz}\,dk.\\
\dot{f}(z,0) &= \mathrm{Re}\left[\dot{\tilde{f}}(z,0)\right]=\frac{1}{2}\left[\dot{\tilde{f}}(z,0)+\dot{\tilde{f}}(z,0)^*\right]=\int_{-\infty}^{\infty}\frac{1}{2}[-i\omega\tilde{A}(k)+i\omega\tilde{A}(-k)^*]e^{ikz}\,dk.
\end{aligned}$$

$\frac{-i\omega}{2}\left[\tilde{A}(k)-\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\dot{f}(z,0)e^{-ikz}\,dz,$ or $\frac{1}{2}\left[\tilde{A}(k)-\tilde{A}(-k)^*\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}\,dz.$

Adding these two results, we get $\boxed{\tilde{A}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f(z,0)+\frac{i}{\omega}\dot{f}(z,0)\right]e^{-ikz}\,dz.}$ qed

Problem 9.33

(a) (i) *Gauss's law:* $\nabla\cdot\mathbf{E}=\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}=0.$ ✓

(ii) *Faraday's law:*

$$\begin{aligned}
-\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{E}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\,\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\,\hat{\boldsymbol{\theta}}\\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[E_0\frac{\sin^2\theta}{r}\left(\cos u-\frac{1}{kr}\sin u\right)\right]\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}\left[E_0\sin\theta\left(\cos u-\frac{1}{kr}\sin u\right)\right]\hat{\boldsymbol{\theta}}.\\
&\quad\text{But } \frac{\partial}{\partial r}\cos u=-k\sin u;\ \frac{\partial}{\partial r}\sin u=k\cos u.\\
&= \frac{1}{r\sin\theta}\frac{E_0}{r}2\sin\theta\cos\theta\left(\cos u-\frac{1}{kr}\sin u\right)\hat{\mathbf{r}}-\frac{1}{r}E_0\sin\theta\left(-k\sin u+\frac{1}{kr^2}\sin u-\frac{1}{r}\cos u\right)\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Integrating with respect to t, and noting that $\int\cos u\,dt=-\frac{1}{\omega}\sin u$ and $\int\sin u\,dt=\frac{1}{\omega}\cos u$, we obtain

$$\boxed{\mathbf{B}=\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u+\frac{1}{kr}\cos u\right)\hat{\mathbf{r}}+\frac{E_0\sin\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)\hat{\boldsymbol{\theta}}.}$$

(iii) *Divergence of* $\mathbf{B}$:

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2E_0\cos\theta}{\omega}\left(\sin u+\frac{1}{kr}\cos u\right)\right]+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{E_0\sin^2\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)\right]\\
&= \frac{1}{r^2}\frac{2E_0\cos\theta}{\omega}\left(k\cos u-\frac{1}{kr^2}\cos u-\frac{1}{r}\sin u\right)\\
&\quad+\frac{1}{r\sin\theta}\frac{2E_0\sin\theta\cos\theta}{\omega r}\left(-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right)
\end{aligned}$$

$$= \frac{2E_0\cos\theta}{\omega r^2}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right) = 0. \checkmark$$

(iv) *Ampére/Maxwell:*

$$\begin{aligned}
\nabla\times\mathbf{B} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{E_0\sin\theta}{\omega}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] - \frac{\partial}{\partial\theta}\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\right]\right\}\hat{\boldsymbol{\phi}} \\
&= \frac{E_0\sin\theta}{\omega r}\left(k^2\sin u - \frac{2}{kr^3}\cos u - \frac{1}{r^2}\sin u - \frac{1}{r^2}\sin u + \frac{k}{r}\cos u + \frac{2}{r^2}\sin u + \frac{2}{kr^3}\cos u\right)\hat{\boldsymbol{\phi}} \\
&= \frac{k}{\omega}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\boldsymbol{\phi}} = \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\boldsymbol{\phi}}. \\
\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \frac{1}{c^2}\frac{E_0\sin\theta}{r}\left(\omega\sin u + \frac{\omega}{kr}\cos u\right)\hat{\boldsymbol{\phi}} = \frac{1}{c^2}\frac{\omega}{k}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\boldsymbol{\phi}} \\
&= \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\boldsymbol{\phi}} = \nabla\times\mathbf{B}. \checkmark
\end{aligned}$$

(b) *Poynting Vector:*

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{E_0\sin\theta}{\mu_0 r}\left(\cos u - \frac{1}{kr}\sin u\right)\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\boldsymbol{\theta}}\right. \\
&\quad \left. + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)(-\hat{\mathbf{r}})\right] \\
&= \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u) - \frac{1}{k^2r^2}\sin u\cos u\right]\hat{\boldsymbol{\theta}}\right. \\
&\quad \left. - \sin\theta\left(-k\cos^2 u + \frac{1}{kr^2}\cos^2 u + \frac{1}{r}\sin u\cos u + \frac{1}{r}\sin u\cos u - \frac{1}{k^2r^3}\sin u\cos u - \frac{1}{kr^2}\sin^2 u\right)\hat{\mathbf{r}}\right\} \\
&= \boxed{\begin{aligned}&\frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u)\right]\hat{\boldsymbol{\theta}}\right. \\
&\qquad \left. + \sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}(\sin^2 u - \cos^2 u)\right]\hat{\mathbf{r}}\right\}.\end{aligned}}
\end{aligned}$$

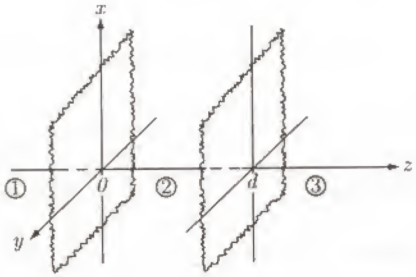
Averaging over a full cycle, using $\langle\sin u\cos u\rangle = 0$, $\langle\sin^2 u\rangle = \langle\cos^2 u\rangle = \frac{1}{2}$, we get the intensity:

$$\mathbf{I} = \langle\mathbf{S}\rangle = \frac{E_0^2\sin\theta}{\mu_0\omega r^2}\left(\frac{k}{2}\sin\theta\right)\hat{\mathbf{r}} = \boxed{\frac{E_0^2\sin^2\theta}{2\mu_0 c r^2}\hat{\mathbf{r}}.}$$

It points in the $\hat{\mathbf{r}}$ direction, and falls off as $1/r^2$, as we would expect for a spherical wave.

(c) $P = \int \mathbf{I}\cdot d\mathbf{a} = \frac{E_0^2}{2\mu_0 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{E_0^2}{2\mu_0 c}2\pi\int_0^\pi\sin^3\theta\,d\theta = \boxed{\frac{4\pi}{3}\frac{E_0^2}{\mu_0 c}.}$

Problem 9.34



$$z<0: \quad \begin{cases} \tilde{\mathbf{E}}_I(z,t)=\tilde{E}_I e^{i(k_1z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_I(z,t)=\frac{1}{v_1}\tilde{E}_I e^{i(k_1z-\omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_R(z,t)=\tilde{E}_R e^{i(-k_1z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_R(z,t)=-\frac{1}{v_1}\tilde{E}_R e^{i(-k_1z-\omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$0<z<d: \quad \begin{cases} \tilde{\mathbf{E}}_r(z,t)=\tilde{E}_r e^{i(k_2z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_r(z,t)=\frac{1}{v_2}\tilde{E}_r e^{i(k_2z-\omega t)}\,\hat{\mathbf{y}} \\ \tilde{\mathbf{E}}_l(z,t)=\tilde{E}_l e^{i(-k_2z-\omega t)}\,\hat{\mathbf{x}}, & \tilde{\mathbf{B}}_l(z,t)=-\frac{1}{v_2}\tilde{E}_l e^{i(-k_2z-\omega t)}\,\hat{\mathbf{y}}. \end{cases}$$

$$z>d: \quad \Big\{ \tilde{\mathbf{E}}_T(z,t)=\tilde{E}_T e^{i(k_3z-\omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t)=\frac{1}{v_3}\tilde{E}_T e^{i(k_3z-\omega t)}\,\hat{\mathbf{y}}.$$

Boundary conditions: $\mathbf{E}_1^{\parallel}=\mathbf{E}_2^{\parallel}$, $\mathbf{B}_1^{\parallel}=\mathbf{B}_2^{\parallel}$, at each boundary (assuming $\mu_1=\mu_2=\mu_3=\mu_0$):

$$z=0: \quad \begin{cases} \tilde{E}_I+\tilde{E}_R=\tilde{E}_r+\tilde{E}_l; \\ \\ \dfrac{1}{v_1}\tilde{E}_I-\dfrac{1}{v_1}\tilde{E}_R=\dfrac{1}{v_2}\tilde{E}_r-\dfrac{1}{v_2}\tilde{E}_l \Rightarrow \tilde{E}_I-\tilde{E}_R=\beta(\tilde{E}_r-\tilde{E}_l), \text{ where } \beta\equiv v_1/v_2. \end{cases}$$

$$z=d: \quad \begin{cases} \tilde{E}_r e^{ik_2d}+\tilde{E}_l e^{-ik_2d}=\tilde{E}_T e^{ik_3d}; \\ \\ \dfrac{1}{v_2}\tilde{E}_r e^{ik_2d}-\dfrac{1}{v_2}\tilde{E}_l e^{-ik_2d}=\dfrac{1}{v_3}\tilde{E}_T e^{ik_3d} \Rightarrow \tilde{E}_r e^{ik_2d}-\tilde{E}_l e^{-ik_2d}=\alpha\tilde{E}_T e^{ik_3d}, \text{ where } \alpha\equiv v_2/v_3. \end{cases}$$

We have here four equations; the problem is to eliminate $\tilde{E}_R$, $\tilde{E}_r$, and $\tilde{E}_l$, to obtain a single equation for $\tilde{E}_T$ in terms of $\tilde{E}_I$.

Add the first two to eliminate $\tilde{E}_R$: $\quad 2\tilde{E}_I=(1+\beta)\tilde{E}_r+(1-\beta)\tilde{E}_l;$
Add the last two to eliminate $\tilde{E}_l$: $\quad 2\tilde{E}_r e^{ik_2d}=(1+\alpha)\tilde{E}_T e^{ik_3d};$
Subtract the last two to eliminate $\tilde{E}_r$: $\quad 2\tilde{E}_l e^{-ik_2d}=(1-\alpha)\tilde{E}^T e^{ik_3d}.$

Plug the last two of these into the first:

$$\begin{aligned} 2\tilde{E}_I &= (1+\beta)\frac{1}{2}e^{-ik_2d}(1+\alpha)\tilde{E}_T e^{ik_3d}+(1-\beta)\frac{1}{2}e^{ik_2d}(1-\alpha)\tilde{E}_T e^{ik_3d} \\ 4\tilde{E}_I &= \left[(1+\alpha)(1+\beta)e^{-ik_2d}+(1-\alpha)(1-\beta)e^{ik_2d}\right]\tilde{E}_T e^{ik_3d} \\ &= \left[(1+\alpha\beta)\left(e^{-ik_2d}+e^{ik_2d}\right)+(\alpha+\beta)\left(e^{-ik_2d}-e^{ik_2d}\right)\right]\tilde{E}_T e^{ik_3d} \\ &= 2\left[(1+\alpha\beta)\cos(k_2d)-i(\alpha+\beta)\sin(k_2d)\right]\tilde{E}_T e^{ik_3d}. \end{aligned}$$

Now the transmission coefficient is $T = \frac{v_3 \epsilon_3 E_{T_0}^2}{v_1 \epsilon_1 E_{I_0}^2} = \frac{v_3}{v_1}\left(\frac{\mu_0 \epsilon_3}{\mu_0 \epsilon_1}\right)\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \frac{v_1}{v_3}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \alpha\beta\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2}$, so

$$\begin{aligned}
T^{-1} &= \frac{1}{\alpha\beta}\frac{|\tilde{E}_I|^2}{|\tilde{E}_T|^2} = \frac{1}{\alpha\beta}\left|\frac{1}{2}\left[(1+\alpha\beta)\cos(k_2 d) - i(\alpha+\beta)\sin(k_2 d)\right]e^{ik_3 d}\right|^2 \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2\cos^2(k_2 d) + (\alpha+\beta)^2\sin^2(k_2 d)\right]. \quad \text{But } \cos^2(k_2 d) = 1 - \sin^2(k_2 d). \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 + (\alpha^2 + 2\alpha\beta + \beta^2 - 1 - 2\alpha\beta - \alpha^2\beta^2)\sin^2(k_2 d)\right] \\
&= \frac{1}{4\alpha\beta}\left[(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)\right]. \\
&\ \text{But } n_1 = \frac{c}{v_1},\ n_2 = \frac{c}{v_2},\ n_3 = \frac{c}{v_3},\ \text{so } \alpha = \frac{n_3}{n_3},\ \beta = \frac{n_2}{n_1}. \\
&= \boxed{\frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2 - n_2^2)(n_3^2 - n_2^2)}{n_2^2}\sin^2(k_2 d)\right].}
\end{aligned}$$

Problem 9.35

$T = 1 \Rightarrow \sin kd = 0 \Rightarrow kd = 0, \pi, 2\pi \ldots.$ The *minimum* (nonzero) thickness is $d = \pi/k$. But $k = \omega/v = 2\pi\nu/v = 2\pi\nu n/c$, and $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ (Eq. 9.69), where (presumably) $\mu \approx \mu_0$. So $n = \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$, and hence $d = \frac{\pi c}{2\pi\nu\sqrt{\epsilon_r}} = \frac{c}{2\nu\sqrt{\epsilon_r}} = \frac{3\times 10^8}{2(10\times 10^9)\sqrt{2.5}} = 9.49\times 10^{-3}\,\text{m}$, or $\boxed{9.5\,\text{mm.}}$

Problem 9.36

From Eq. 9.199,

$$\begin{aligned}
T^{-1} &= \frac{1}{4(4/3)(1)}\left\{[(4/3)+1]^2 + \frac{[(16/9)-(9/4)][1-(9/4)]}{(9/4)}\sin^2(3\omega d/2c)\right\} \\
&= \frac{3}{16}\left[\frac{49}{9} + \frac{(-17/36)(-5/4)}{(9/4)}\sin^2(3\omega d/2c)\right] = \frac{49}{48} + \frac{85}{(48)(36)}\sin^2(3\omega d/2c). \\
T &= \frac{48}{49 + (85/36)\sin^2(3\omega d/2c)}.
\end{aligned}$$

Since $\sin^2(3\omega d/2c)$ ranges from 0 to 1, $T_{\min} = \frac{48}{49 + (85/36)} = \boxed{0.935;}$ $T_{\max} = \frac{48}{49} = \boxed{0.980.}$ Not much variation, and the transmission is good (over 90%) for *all* frequencies. Since Eq. 9.199 is unchanged when you switch 1 and 3, the transmission is the same either direction, and the $\boxed{\text{fish sees you just as well as you see it.}}$

Problem 9.37

(a) Equation 9.91 $\Rightarrow \tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r} - \omega t)}$; $\mathbf{k}_T\cdot\mathbf{r} = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = k_T(x\sin\theta_T + z\cos\theta_T) = xk_T\sin\theta_T + izk_T\sqrt{\sin^2\theta_T - 1} = kx + i\kappa z$, where

$$\begin{aligned}
k &\equiv k_T\sin\theta_T = \left(\frac{\omega n_2}{c}\right)\frac{n_1}{n_2}\sin\theta_I = \frac{\omega n_1}{c}\sin\theta_I, \\
\kappa &\equiv k_T\sqrt{\sin^2\theta_T - 1} = \frac{\omega n_2}{c}\sqrt{(n_1/n_2)^2\sin^2\theta_I - 1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}. \quad \text{So} \\
\tilde{\mathbf{E}}_T(\mathbf{r},t) &= \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx - \omega t)}. \quad \text{qed}
\end{aligned}$$

(b) $R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$. Here β is real (Eq. 9.106) and α is purely imaginary (Eq. 9.108); write $\alpha = ia$, with a real: $R = \left(\frac{ia-\beta}{ia+\beta}\right)\left(\frac{-ia-\beta}{-ia+\beta}\right) = \frac{a^2+\beta^2}{a^2+\beta^2} = \boxed{1.}$

(c) From Prob. 9.16, $E_{0_R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}$, so $R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = \left|\frac{1-ia\beta}{1+ia\beta}\right|^2 = \frac{(1-ia\beta)(1+ia\beta)}{(1+ia\beta)(1-ia\beta)} = \boxed{1.}$

(d) From the solution to Prob. 9.16, the transmitted wave is

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{z}}).$$

Using the results in (a): $\mathbf{k}_T\cdot\mathbf{r} = kx + i\kappa z - \omega t$, $\sin\theta_T = \frac{ck}{\omega n_2}$, $\cos\theta_T = i\frac{c\kappa}{\omega n_2}$:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\left(-i\frac{c\kappa}{\omega n_2}\,\hat{\mathbf{x}} + \frac{ck}{\omega n_2}\,\hat{\mathbf{z}}\right).$$

We may as well choose the phase constant so that $\tilde{E}_{0_T}$ is *real*. Then

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{y}};\\
\mathbf{B}(\mathbf{r},t) &= \frac{1}{v_2}E_0 e^{-\kappa z}\frac{c}{\omega n_2}\mathrm{Re}\left\{[\cos(kx-\omega t) + i\sin(kx-\omega t)][-i\kappa\,\hat{\mathbf{x}} + k\,\hat{\mathbf{z}}]\right\}\\
&= \frac{1}{\omega}E_0 e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right]. \quad \text{qed}
\end{aligned}$$

(I used $v_2 = c/n_2$ to simplfy $\mathbf{B}$.)

$$\begin{aligned}
\text{(e)}\quad \text{(i) } \nabla\cdot\mathbf{E} &= \frac{\partial}{\partial y}\left[E_0 e^{-\kappa z}\cos(kx-\omega t)\right] = 0.\ \checkmark\\
\text{(ii) } \nabla\cdot\mathbf{B} &= \frac{\partial}{\partial x}\left[\frac{E_0}{\omega}e^{-\kappa z}\kappa\sin(kx-\omega t)\right] + \frac{\partial}{\partial z}\left[\frac{E_0}{\omega}e^{-\kappa z}k\cos(kx-\omega t)\right]\\
&= \frac{E_0}{\omega}\left[e^{-\kappa z}\kappa k\cos(kx-\omega t) - \kappa e^{-\kappa z}k\cos(kx-\omega t)\right] = 0.\ \checkmark\\
\text{(iii) } \nabla\times\mathbf{E} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & E_y & 0\end{vmatrix} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}} + \frac{\partial E_y}{\partial x}\,\hat{\mathbf{z}}\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - E_0 e^{-\kappa z}k\sin(kx-\omega t)\,\hat{\mathbf{z}}.\\
-\frac{\partial\mathbf{B}}{\partial t} &= -\frac{E_0}{\omega}e^{-\kappa z}\left[-\kappa\omega\cos(kx-\omega t)\,\hat{\mathbf{x}} + k\omega\sin(kx-\omega t)\,\hat{\mathbf{z}}\right]\\
&= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - kE_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{z}} = \nabla\times\mathbf{E}.\ \checkmark\\
\text{(iv) } \nabla\times\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ B_x & 0 & B_z\end{vmatrix} = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{\mathbf{y}}\\
&= \left[-\frac{E_0}{\omega}\kappa^2 e^{-\kappa z}\sin(kx-\omega t) + \frac{E_0}{\omega}e^{-\kappa z}k^2\sin(kx-\omega t)\right]\hat{\mathbf{y}} = (k^2-\kappa^2)\frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.
\end{aligned}$$

$$\text{Eq. 9.202} \Rightarrow k^2 - \kappa^2 = \left(\frac{\omega}{c}\right)^2\left[n_1^2\sin^2\theta_I - (n_1\sin\theta_I)^2 + (n_2)^2\right] = \left(\frac{n_2\omega}{c}\right)^2 = \omega^2\epsilon_2\mu_2.$$

$$\begin{aligned} &= \epsilon_2\mu_2\omega E_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.\\ \mu_2\epsilon_2\frac{\partial \mathbf{E}}{\partial t} &= \mu_2\epsilon_2 E_0 e^{-\kappa z}\omega\sin(kx-\omega t)\,\hat{\mathbf{y}} = \boldsymbol{\nabla}\times\mathbf{B}\ \checkmark.\end{aligned}$$

(f)

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_2}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_2}\frac{E_0^2}{\omega}e^{-2\kappa z}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & \cos(kx-\omega t) & 0\\ \kappa\sin(kx-\omega t) & 0 & k\cos(kx-\omega t)\end{vmatrix}\\ &= \boxed{\frac{E_0^2}{\mu_2\omega}e^{-2\kappa z}\left[k\cos^2(kx-\omega t)\,\hat{\mathbf{x}} - \kappa\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].}\end{aligned}$$

Averaging over a complete cycle, using $\langle\cos^2\rangle = 1/2$ and $\langle\sin\cos\rangle = 0$, $\langle\mathbf{S}\rangle = \dfrac{E_0^2 k}{2\mu_2\omega}e^{-2\kappa z}\,\hat{\mathbf{x}}$. On average, then, no energy is transmitted in the z direction, only in the x direction (parallel to the interface). qed

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Problem 9.38

Look for solutions of the form $\mathbf{E} = \mathbf{E}_0(x,y,z)e^{-i\omega t}$, $\mathbf{B} = \mathbf{B}_0(x,y,z)e^{-i\omega t}$, subject to the boundary conditions $\mathbf{E}^{\parallel} = 0$, $B^{\perp} = 0$ at all surfaces. Maxwell's equations, in the form of Eq. 9.177, give

$$\left\{\begin{array}{llll} \boldsymbol{\nabla}\cdot\mathbf{E} = 0 & \Rightarrow \boldsymbol{\nabla}\cdot\mathbf{E}_0 = 0; & \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & \Rightarrow \boldsymbol{\nabla}\times\mathbf{E}_0 = i\omega\mathbf{B}_0;\\ \boldsymbol{\nabla}\cdot\mathbf{B} = 0 & \Rightarrow \boldsymbol{\nabla}\cdot\mathbf{B}_0 = 0; & \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} & \Rightarrow \boldsymbol{\nabla}\times\mathbf{B}_0 = -\frac{i\omega}{c^2}\mathbf{E}_0.\end{array}\right\}$$

From now on I'll leave off the subscript (0). The problem is to solve the (time independent) equations

$$\left\{\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{E} = 0; & \boldsymbol{\nabla}\times\mathbf{E} = i\omega\mathbf{B};\\ \boldsymbol{\nabla}\cdot\mathbf{B} = 0; & \boldsymbol{\nabla}\times\mathbf{B} = -\frac{i\omega}{c^2}\mathbf{E}.\end{array}\right\}$$

From $\boldsymbol{\nabla}\times\mathbf{E} = i\omega\mathbf{B}$ it follows that I can get $\mathbf{B}$ once I know $\mathbf{E}$, so I'll concentrate on the latter for the moment.

$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = \boldsymbol{\nabla}\times(i\omega\mathbf{B}) = i\omega\left(-\dfrac{i\omega}{c^2}\mathbf{E}\right) = \dfrac{\omega^2}{c^2}\mathbf{E}$. So $\nabla^2 E_x = -\left(\dfrac{\omega}{c}\right)^2 E_x$; $\nabla^2 E_y = -\left(\dfrac{\omega}{c}\right)^2 E_y$; $\nabla^2 E_z = -\left(\dfrac{\omega}{c}\right)^2 E_z$. Solve each of these by separation of variables: $E_x(x,y,z) = X(x)Y(y)Z(z) \Rightarrow YZ\dfrac{d^2X}{dx^2} + ZX\dfrac{d^2Y}{dy^2} + XY\dfrac{d^2Z}{dz^2} = -\left(\dfrac{\omega}{c}\right)^2 XYZ$, or $\dfrac{1}{X}\dfrac{d^2X}{dx^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} + \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = -(\omega/c)^2$. Each term must be a constant, so $\dfrac{d^2X}{dx^2} = -k_x^2 X$, $\dfrac{d^2Y}{dy^2} = -k_y^2 Y$, $\dfrac{d^2Z}{dz^2} = -k_z^2 Z$, with $k_x^2 + k_y^2 + k_z^2 = -(\omega/c)^2$. The solution is

$$E_x(x,y,z) = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)][E\sin(k_z z) + F\cos(k_z z)].$$

But $\mathbf{E}^{\parallel} = 0$ at the boundaries $\Rightarrow E_x = 0$ at $y = 0$ and $z = 0$, so $D = F = 0$, and $E_x = 0$ at $y = b$ and $z = d$, so $k_y = n\pi/b$ and $k_z = l\pi/d$, where n and l are integers. A similar argument applies to E_y and E_z. *Conclusion:*

$$\begin{aligned} E_x(x,y,z) &= [A\sin(k_x x) + B\cos(k_x x)]\sin(k_y y)\sin(k_z z),\\ E_y(x,y,z) &= \sin(k_x x)[C\sin(k_y y) + D\cos(k_y y)]\sin(k_z z),\\ E_z(x,y,z) &= \sin(k_x x)\sin(k_y y)[E\sin(k_z z) + F\cos(k_z z)],\end{aligned}$$

where $k_x = m\pi/a$. (Actually, there is no reason at this stage to assume that k_x, k_y, and k_z are the *same* for all three components, and I should really affix a second subscript (x for E_x, y for E_y, and z for E_z), but in a moment we shall see that *in fact* they *do* have to be the same, so to avoid cumbersome notation I'll *assume* they are from the start.)

Now $\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \Rightarrow k_x[A\cos(k_x x) - B\sin(k_x x)]\sin(k_y y)\sin(k_z z) + k_y\sin(k_x x)[C\cos(k_y y) - D\sin(k_y y)]\sin(k_z z) + k_z\sin(k_x x)\sin(k_y y)[E\cos(k_z z) - F\sin(k_z z)] = 0$. In particular, putting in $x = 0$, $k_x A\sin(k_y y)\sin(k_z z) = 0$, and hence $A = 0$. Likewise $y = 0 \Rightarrow C = 0$ and $z = 0 \Rightarrow E = 0$. (Moreover, if the k's were *not* equal for different components, then by Fourier analysis this equation could not be satisfied (for all x, y, and z) unless the other three constants were *also* zero, and we'd be left with no field at all.) It follows that $-(Bk_x + Dk_y + Fk_z) = 0$ (in order that $\nabla \cdot \mathbf{E} = 0$), and we are left with

$$\boxed{\begin{array}{l}\mathbf{E} = B\cos(k_x x)\sin(k_y y)\sin(k_z z)\,\hat{\mathbf{x}} + D\sin(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{y}} + F\sin(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{z}},\\ \text{with } k_x = (m\pi/a),\ k_y = (n\pi/b),\ k_z = (l\pi/d)\ (l,\ m,\ n \text{ all integers}),\ \text{and } Bk_x + Dk_y + Fk_z = 0.\end{array}}$$

The corresponding magnetic field is given by $\mathbf{B} = -(i/\omega)\nabla \times \mathbf{E}$:

$$\begin{aligned}
B_x &= -\frac{i}{\omega}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{i}{\omega}\left[Fk_y\sin(k_x x)\cos(k_y y)\cos(k_z z) - Dk_z\sin(k_x x)\cos(k_y y)\cos(k_z z)\right],\\
B_y &= -\frac{i}{\omega}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{i}{\omega}\left[Bk_z\cos(k_x x)\sin(k_y y)\cos(k_z z) - Fk_x\cos(k_x x)\sin(k_y y)\cos(k_z z)\right],\\
B_z &= -\frac{i}{\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{i}{\omega}\left[Dk_x\cos(k_x x)\cos(k_y y)\sin(k_z z) - Bk_y\cos(k_x x)\cos(k_y y)\sin(k_z z)\right].
\end{aligned}$$

Or:

$$\boxed{\begin{aligned}\mathbf{B} &= -\frac{i}{\omega}(Fk_y - Dk_z)\sin(k_x x)\cos(k_y y)\cos(k_z z)\,\hat{\mathbf{x}} - \frac{i}{\omega}(Bk_z - Fk_x)\cos(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{y}}\\ &\quad - \frac{i}{\omega}(Dk_x - Bk_y)\cos(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{z}}.\end{aligned}}$$

These *automatically* satisfy the boundary condition $B^{\perp} = 0$ ($B_x = 0$ at $x = 0$ and $x = a$, $B_y = 0$ at $y = 0$ and $y = b$, and $B_z = 0$ at $z = 0$ and $z = d$).

As a check, let's see if $\nabla \cdot \mathbf{B} = 0$:

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= -\frac{i}{\omega}(Fk_y - Dk_z)k_x\cos(k_x x)\cos(k_y y)\cos(k_z z) - \frac{i}{\omega}(Bk_z - Fk_x)k_y\cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&\quad - \frac{i}{\omega}(Dk_x - Bk_y)k_z\cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&= -\frac{i}{\omega}(Fk_xk_y - Dk_xk_z + Bk_zk_y - Fk_xk_y + Dk_xk_z - Bk_yk_z)\cos(k_x x)\cos(k_y y)\cos(k_z z) = 0.\ \checkmark
\end{aligned}$$

The boxed equations satisfy all of Maxwell's equations, and they meet the boundary conditions. For TE modes, we pick $E_z = 0$, so $F = 0$ (and hence $Bk_x + Dk_y = 0$, leaving only the overall amplitude undetermined, for given l, m, and n); for TM modes we want $B_z = 0$ (so $Dk_x - Bk_y = 0$, again leaving only one amplitude undetermined, since $Bk_x + Dk_y + Fk_z = 0$). In either case (TE_{lmn} or TM_{lmn}), the frequency is given by $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = c^2\left[(m\pi/a)^2 + (n\pi/b)^2 + (l\pi/d)^2\right]$, or $\boxed{\omega = c\pi\sqrt{(m/a)^2 + (n/b)^2 + (l/d)^2}.}$